Problem 24:

1) $Z(\mu', T) = \mathrm{Tr}\, e^{-\beta\hat{H} + \beta\mu'\hat{N}}$

$$= \sum_{\substack{\{n_{\vec{k},i}\} \\ i = 0 \text{ or } 1}} \prod_{\vec{k}} e^{-\beta\varepsilon_{\vec{k}} n_{\vec{k},0} + \beta\mu' n_{\vec{k},0}}\, e^{-\beta(\varepsilon_{\vec{k}}+\Delta) n_{\vec{k},1} + \beta\mu' n_{\vec{k},1}}$$

$$= \prod_{\vec{k}} \underbrace{\left(\sum_{n=0}^{\infty} e^{-\beta\varepsilon_{\vec{k}} n + \beta\mu' n}\right)}_{\frac{1}{1-e^{-\beta(\varepsilon_{\vec{k}}-\mu')}}} \underbrace{\left(\sum_{n=0}^{\infty} e^{-\beta(\varepsilon_{\vec{k}}+\Delta)n + \beta\mu' n}\right)}_{\frac{1}{1-e^{-\beta(\varepsilon_{\vec{k}}+\Delta-\mu')}}}$$

$$\Rightarrow \Omega = -k_B T \sum_{\vec{k}} \ln\left(1 - e^{-\beta(\varepsilon_{\vec{k}}-\mu')}\right) - k_B T \sum_{\vec{k}} \left(1 - e^{-\beta(\varepsilon_{\vec{k}}+\Delta-\mu')}\right)$$

$$= \Omega_0(\mu', T) + \Omega_0(\mu' - \Delta, T) \quad (2)$$

if $\Omega_0(\mu', T)$ is grand potential of "conventional" ideal Bose-Einstein gas

for Ω_0: $\langle n\rangle = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z)$ $\qquad z = e^{\beta\mu}$

$z' = e^{\beta(\mu'-\Delta)}$

here: $$\langle n\rangle = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z) + \frac{1}{V}\frac{z'}{1-z'} + \frac{1}{\lambda_T^3} g_{3/2}(z')$$

$$= \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z) + \underbrace{\frac{1}{V}\frac{z e^{-\beta\Delta}}{1-z e^{-\beta\Delta}}}_{\xrightarrow{V\to\infty} 0 \text{ even for } z=1} + \frac{1}{\lambda_T^3} g_{3/2}(z e^{-\beta\Delta}) \quad (2)$$

maximal density in excited states at $z = z_{max} = 1$

$$\langle n\rangle_{max} = \frac{1}{\lambda_T^3} g_{3/2}(1) + \frac{1}{\lambda_T^3} g_{3/2}(e^{-\beta\Delta})$$

$\Rightarrow$ transition at $$n = \frac{1}{\lambda_{T_c}^3} g_{3/2}(1) + \frac{1}{\lambda_{T_c}^3} g_{3/2}(e^{-\beta\Delta})$$

(1)

if $\beta\Delta \gg 1 \Rightarrow e^{-\beta\Delta} \ll 1 \Rightarrow g_{3/2}(e^{-\beta\Delta}) \approx e^{-\beta\Delta}$

$$\lambda_{T_c}^3 n = g_{3/2}(1) + e^{-\frac{\Delta}{k_B T_c}}$$

1 (*)

$$n\left(\frac{h^2}{2\pi m k_B T_c}\right)^{3/2}$$ ← equation for T_c

e) ~~At~~ ~~also~~ Without the second state, we would have

$$\lambda_{T_c}^3 n = g_{3/2}(1)$$

With the second state the right hand side of (*) is larger than without the second state

$\Rightarrow T_c$ is lower

2

Problem 25:

a) one energy per $\vec{l}$, $\varepsilon = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \vec{l}^{\,2}$

$\to$ number of $\vec{l}$'s below ε_F is

$$\frac{4}{3}\pi\left(\frac{2mL^2\varepsilon_F}{h^2}\right)^{3/2}$$

with spin $= \frac{1}{2}$ this has to equal $\frac{N}{2}$

$$\frac{N}{2} = \frac{4}{3}\pi\left(\frac{2m\,\varepsilon_F}{h^2}\right)^{3/2} V$$

$$\frac{\hbar^2}{2m}\left(\frac{3}{8\pi}\frac{N}{V}\right)^{2/3} = \varepsilon_F \quad 2$$

$$\frac{N}{V} = 9\,\frac{g}{cm^3}\,\frac{1}{63.5\,\frac{g}{mol}}\cdot 6\cdot 10^{23}\,\frac{1}{mol} = 8.5\cdot 10^{28}\,\frac{1}{m^3} \quad 2$$

$$\varepsilon_F = 1.13\cdot 10^{-18}\,J = 7\,eV \quad 1$$

b) $\cancel{T_F = \frac{k_B}{1.13\cdot 10^{-18}\,J} = 1.2\cdot 10^{25}\,K}$

$$T_F = \frac{1.13\cdot 10^{-18}\,J}{k_B} = 8.2\cdot 10^4\,K \quad 1$$

Problem 26:

a) $\Omega(\varepsilon) =$ # states with energy less or equal ε

$= \#\ \vec{l}\text{'s with } \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \vec{l}^{\,2} \le \varepsilon$

$$= \pi\left(\frac{2m\varepsilon}{\hbar^2}\frac{L^2}{(2\pi)^2}\right) = \frac{2\pi m}{h^2} A\varepsilon$$

$$\Rightarrow n(\varepsilon) = \frac{2\pi m}{h^2} A \quad 2$$

b)

$$N = 2\int_0^{\varepsilon_F} n(\varepsilon)\, d\varepsilon = \frac{4\pi m}{h^2} A \varepsilon_F \quad 1 \quad \Rightarrow \quad \varepsilon_F = \frac{h^2}{4\pi m}\frac{N}{A} = \frac{h^2}{4\pi m} n \quad 1$$

$$n(\varepsilon) = \frac{N}{2\varepsilon_F} \quad 1$$

c)

$$N = \int_{-\infty}^{\infty} n(\varepsilon) \frac{2}{e^{\beta(\varepsilon-\mu')}+1} d\varepsilon \quad 1$$

$$= \int_{-\infty}^{\infty} \Omega(\varepsilon) \frac{2\beta e^{\beta(\varepsilon-\mu')}}{(e^{\beta(\varepsilon-\mu')}+1)^2} d\varepsilon$$

$$\underset{t=\beta(\varepsilon-\mu')}{=} \int_{-\infty}^{\infty} \Omega(\mu' + k_B T t) \frac{2e^t}{(e^t+1)^2} dt \quad 1$$

$$= \int_{-\infty}^{\infty} \frac{N}{2}\frac{\mu' + k_B T t}{\varepsilon_F} \frac{2e^t}{(e^t+1)^2} dt$$

$$= \frac{N}{2} \cdot 2\frac{\mu'}{\varepsilon_F} \quad \Rightarrow \quad \mu' = \varepsilon_F \quad 1$$

d) $U = \int_{-\infty}^{\infty} n(\varepsilon)\, \varepsilon \, \frac{2}{e^{\beta(\varepsilon-\mu')}+1} \, d\varepsilon$ ①

$$= \frac{N}{2\varepsilon_F} \cdot 2 \int_{-\infty}^{\infty} \varepsilon \, \frac{1}{e^{\beta(\varepsilon-\mu')}+1} \, d\varepsilon$$

$$= \frac{N}{\varepsilon_F} \int_{-\infty}^{\infty} \frac{\varepsilon^2}{2} \, \frac{e^{\beta(\varepsilon-\mu')}}{\left(e^{\beta(\varepsilon-\mu')}+1\right)^2} \, d\beta\varepsilon$$

$$= \frac{N}{2\varepsilon_F} \int_{-\infty}^{\infty} (\mu' + k_B T t)^2 \, \frac{e^{t}}{(e^{t}+1)^2} \, dt$$

$$= \frac{N}{2\varepsilon_F} \varepsilon_F^{\,2} + \frac{N}{2\varepsilon_F} (k_B T)^2 \frac{\pi^2}{3}$$

$$= \frac{N}{2} \varepsilon_F + \frac{N\pi^2}{6} \varepsilon_F \left(\frac{k_B T}{\varepsilon_F}\right)^2$$ ①