Problem 21: also accept periodic boundary condition

a) $$\hat\rho = \frac{e^{-\beta \hat H}}{Tr\, e^{-\beta \hat H}}$$

$$Tr\, e^{-\beta\hat H} = \sum_{\vec k} e^{-\beta\frac{\hbar^2 k^2}{2m}} = \sum_{n_x,n_y,n_z\ge 1} e^{-\beta\frac{\hbar^2\pi^2}{2mL^2}(n_x^2+n_y^2+n_z^2)}$$

$$\approx \int_{n_x,n_y,n_z\ge 0} d^3\vec n\, e^{-\beta\frac{\hbar^2\pi^2}{2mL^2}\vec n^2} = \frac{1}{8}\left(\frac{2\pi m L^2 k_B T}{\hbar^2\pi^2}\right)^{3/2} = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

$$= \frac{V}{\lambda_T^3}$$ 2

$$\langle \vec k'|\hat\rho|\vec k\rangle = \frac{\lambda_T^3}{V}\langle\vec k'|\vec k\rangle e^{-\beta\frac{\hbar^2 k^2}{2m}}$$

$$= \frac{\lambda_T^3}{V} e^{-\beta\frac{\hbar^2 k^2}{2m}}\delta_{\vec k,\vec k'}$$ 2

b) $$\langle\vec r'|\hat\rho|\vec r\rangle = \sum_{\vec k,\vec k'}\langle \vec r'|\vec k'\rangle\langle\vec k'|\hat\rho|\vec k\rangle\langle\vec k|\vec r\rangle$$

$$= \frac{\lambda_T^3}{V}\sum_{\vec k}\langle\vec r'|\vec k\rangle e^{-\beta\frac{\hbar^2 k^2}{2m}}\langle\vec k|\vec r\rangle$$

$$= \frac{\lambda_T^3}{V^2}\sum_{\vec k} e^{-\beta\frac{\hbar^2 k^2}{2m}} e^{i\vec k(\vec r'-\vec r)}$$ 2

$$= \frac{\lambda_T^3}{V^2}\sum_{n_x,n_y,n_z\ge 1} e^{-\beta\frac{\hbar^2\pi^2}{2mL^2}(n_x^2+n_y^2+n_z^2)}\, e^{i\frac{\pi}{L}(\vec r'-\vec r)\begin{pmatrix}n_x\\ n_y\\ n_z\end{pmatrix}}$$

$$\approx \frac{\lambda_T^3}{8V^2}\int d^3\vec n\, e^{-\frac{\beta\hbar^2\pi^2}{2mL^2}\vec n^2}\, e^{i\frac{\pi}{L}(\vec r'-\vec r)\vec n}$$ 2

$$= \frac{\lambda_T^3}{8V^2}\frac{V}{\pi^3}\int d^3\vec n\, e^{-\frac{\beta\hbar^2}{2m}\vec n^2 + i\vec n(\vec r'-\vec r)}$$

$$= \frac{\lambda_T^3}{\text{[illegible]}}\, e^{-\frac{m(\vec r'-\vec r)^2}{2\beta\hbar^2}}\int d^3\vec n\, e^{-\frac{\beta\hbar^2}{2m}\left(\vec n - \frac{m i}{\beta\hbar^2}(\vec r'-\vec r)\right)^2}$$

$$= \frac{\lambda_T^3}{8V\pi^3}\left(\frac{2\pi m k_B T}{\hbar^2}\right)^{3/2} e^{-\frac{m}{2\beta\hbar^2}(\vec{r}-\vec{r}')^2}$$

$$= \frac{1}{V}\, e^{-\frac{m k_B T 4\pi^2}{2h^2}(\vec{r}-\vec{r}')^2} = \frac{1}{V}\, e^{-\frac{\pi}{\lambda_T^2}(\vec{r}-\vec{r}')^2} \quad (2)$$

Problem 22:

a) $p_n = \langle n | \hat{\varrho} | n \rangle = \langle n | \dfrac{e^{-\beta \hat{H} + \beta \mu' \hat{N}}}{Tr\, e^{-\beta \hat{H} + \beta \mu' \hat{N}}} | n \rangle$

$$= \frac{e^{-\beta(\varepsilon - \mu')n}}{\sum_{n=0}^{\infty} e^{-\beta(\varepsilon-\mu')n}} = \left(1 - e^{-\beta(\varepsilon-\mu')}\right) e^{-\beta(\varepsilon-\mu')n}$$

$$= \left(1 - z e^{-\beta\varepsilon}\right)\left(z e^{-\beta\varepsilon}\right)^n$$ 2

b) $$Tr\, \hat{\varrho}\hat{N} = \sum_{n=0}^{\infty} \langle n | \hat{\varrho}\hat{N} | n \rangle = \sum_{n=0}^{\infty} n \left(1 - z e^{-\beta\varepsilon}\right)\left(z e^{-\beta\varepsilon}\right)^n$$ 1

$$= \left(1 - z e^{-\beta\varepsilon}\right) z \frac{d}{dz} \sum_{n=0}^{\infty} \left(z e^{-\beta\varepsilon}\right)^n$$

$$= \left(1 - z e^{-\beta\varepsilon}\right) z \frac{d}{dz} \frac{1}{1 - z e^{-\beta\varepsilon}}$$

$$= \left(1 - z e^{-\beta\varepsilon}\right) z \frac{+e^{-\beta\varepsilon}}{\left(1 - z e^{-\beta\varepsilon}\right)^2} = \frac{z}{e^{\beta\varepsilon} - z}$$ 2

c) $\langle n | \hat{\varrho} | n \rangle = \langle n | \dfrac{e^{-\beta \hat{H} + \beta \mu' \hat{N}}}{Tr\, e^{-\beta \hat{H} + \beta \mu' \hat{N}}} | n \rangle$

$$= \frac{e^{-\beta(\varepsilon-\mu')n}}{\sum_{n=0}^{1} e^{-\beta(\varepsilon-\mu')n}} = \frac{\left(z e^{-\beta\varepsilon}\right)^n}{1 + z e^{-\beta\varepsilon}}$$ 2

d) $$Tr\, \hat{\varrho}\hat{N} = \sum_{n=0}^{1} \langle n | \hat{\varrho}\hat{N} | n \rangle = \frac{0 \cdot \left(z e^{-\beta\varepsilon}\right)^0 + 1 \cdot \left(z e^{-\beta\varepsilon}\right)^1}{\left(1 + z e^{-\beta\varepsilon}\right)} = \frac{z}{e^{\beta\varepsilon} + z}$$ 1 2

Problem 23

a)

$$Z_{cl}(T) = \frac{1}{2} \sum_{n_1, n_2 = 0}^{\infty} e^{-\beta\hbar\omega[n_1+n_2+1]}$$ (1)

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\right]^2 = \frac{e^{-\beta\hbar\omega}}{2(1-e^{-\beta\hbar\omega})^2}$$ (1)

b) Use basis

$$|n_1 n_2\rangle^{(+)} = \frac{1}{\sqrt{2}}(|n_1 n_2\rangle + |n_2 n_1\rangle) \quad \text{for } n_1 \neq n_2$$

$$|n\, n\rangle^{(+)} = |n\, n\rangle \quad \text{for all } n$$

$$Z_B(T) = \frac{1}{2} \sum_{n_1 \neq n_2} {}^{(+)}\langle n_1 n_2| e^{-\beta \hat{H}} |n_1 n_2\rangle + \sum_{n=0}^{\infty} {}^{(+)}\langle n\, n| e^{-\beta\hat{H}} |n\, n\rangle$$ (2)

avoid double counting

$$= \frac{1}{2} \sum_{n_1 \neq n_2} e^{-\beta\hbar\omega(n_1+n_2+1)} + \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+1)}$$

$$= \frac{1}{2}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-\beta\hbar\omega(n_1+n_2+1)} - \sum_{n} e^{-\beta\hbar\omega(2n+1)}\right) + \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+1)}$$

$$= \frac{e^{-\beta\hbar\omega}}{2(1-e^{-\beta\hbar\omega})^2} + \frac{e^{-\beta\hbar\omega}}{2(1-e^{-2\beta\hbar\omega})}$$ (2)

c) Use basis

$$|n_1 n_2\rangle^{(-)} = \frac{1}{\sqrt{2}}(|n_1 n_2\rangle - |n_2 n_1\rangle) \quad \text{for } n_1 \neq n_2$$

$$Z_F(T) = \frac{1}{2} \sum_{n_1 \neq n_2} {}^{(-)}\langle n_1 n_2| e^{-\beta\hat{H}} |n_1 n_2\rangle^{(-)} = \frac{1}{2}\sum_{n_1 \neq n_2} e^{-\beta\hbar\omega(n_1+n_2+1)}$$ (2)

$$= \frac{1}{2}\left(\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} e^{-\beta\hbar\omega(n_1+n_2+1)} - \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(2n+1)}\right)$$

$$= \frac{e^{-\beta\hbar\omega}}{2(1-e^{-\beta\hbar\omega})^2} - \frac{e^{-\beta\hbar\omega}}{2(1-e^{-2\beta\hbar\omega})}$$ (2)