Problem 16:

a) $$Z(\mu';T) \stackrel{1}{=} \sum_{\{n_i\}} e^{\beta\varepsilon n_i + \beta\mu' n_i} = \left(\sum_{n=0}^{1} e^{(\beta\varepsilon+\beta\mu')n}\right)^M$$
$$= \left(1 + e^{\beta\varepsilon+\beta\mu'}\right)^M \quad 2$$

b) $$\Omega = -k_BT\ln Z(\mu';T) = -k_BTM\ln\left(1+e^{\beta\varepsilon+\beta\mu'}\right) \quad 1$$

c) $$N = -\left(\frac{\partial\Omega}{\partial\mu'}\right) = k_BTM\,\frac{\beta e^{\beta\varepsilon+\beta\mu'}}{1+e^{\beta\varepsilon+\beta\mu'}} = \frac{M}{1+e^{-(\beta\varepsilon+\beta\mu')}} \quad 2$$

d) $$\frac{N}{M} = \frac{1}{e^{-\beta(\varepsilon+\mu')}+1} \quad 1$$

ideal gas:

$$F = -Nk_BT - Nk_BT\ln\left[\frac{V}{N}\left(\frac{2\pi m k_BT}{h^2}\right)^{3/2}\right]$$

$$\mu' = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -k_BT - k_BT\ln\left(\frac{V}{N}\left(\frac{2\pi m k_BT}{h^2}\right)^{3/2}\right) + Nk_BT\frac{1}{N}$$

$$\underset{PV=Nk_BT}{\overset{=}{\uparrow}} -k_BT\ln\left[\frac{(k_BT)^{5/2}}{P}\left(\frac{2\pi m}{h^2}\right)^{3/2}\right] \quad 1$$

OK to get zero for last quantity

in equilibrium the chemical potential of the gas and the chemical potential of the atoms on the surface have to be identical

$$\Rightarrow \frac{N}{M} = \frac{1}{e^{-\beta\varepsilon}\frac{(k_BT)^{5/2}}{P}\left(\frac{2\pi m}{h^2}\right)^{3/2}+1} \quad 2$$

Problem 17

a) $\hat{\varrho}$ hermitian

$$\Rightarrow \frac{1}{2}\begin{pmatrix} 1+a_1^* & a_3^* \\ a_2^* & 1-a_1^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+a_1 & a_2 \\ a_3 & 1-a_1 \end{pmatrix}$$

$\Rightarrow$ a_1 real, $a_3 = a_2^*$ (2)

$\mathrm{Tr}\,\hat{\varrho} = 1 \Rightarrow \frac{1}{2}(1+a_1) + \frac{1}{2}(1-a_1) = 1 \Rightarrow 1 = 1$ always fulfilled (1)

$\hat{\varrho}$ positive semidefinite $\Leftrightarrow$ both eigenvalues of $\hat{\varrho}$ non-negative

$$\det \frac{1}{2}\begin{pmatrix} 1+a_1-2\lambda & a_2 \\ a_2^* & 1-a_1-2\lambda \end{pmatrix} = (1+a_1-2\lambda)(1-a_1-2\lambda) - a_2 a_2^* = 0$$

$$4\lambda^2 - 4\lambda + 1 - a_1^2 - |a_2|^2 = 0$$

$$\lambda_{1,2} = +1 \pm \sqrt{1 + \frac{a_1^2 + |a_2|^2 - 1}{4}}$$

both ~~positive~~ non-negative $\Leftrightarrow \frac{a_1^2 + |a_2|^2 - 1}{4} \le 0 \Leftrightarrow a_1^2 + |a_2|^2 \le 1$ (2)

pure state $\Leftrightarrow \lambda_2 = 0 \Leftrightarrow a_1^2 + |a_2|^2 = 1$ (1)

b) $\hat{\varrho}^H = \left(\frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\,e^{-\beta\hat{H}}}\right)^H = \frac{e^{-\beta\hat{H}^H}}{(\mathrm{Tr}\,e^{-\beta\hat{H}})^*} = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\,e^{-\beta\hat{H}}} = \hat{\varrho} \Rightarrow \hat{\varrho}$ hermitian (1)

$|\psi\rangle$ any vector $\quad |\varphi\rangle \equiv e^{-\frac{\beta}{2}\hat{H}}|\psi\rangle$

$$0 \le \langle\varphi|\varphi\rangle = \langle\psi| e^{-\frac{\beta}{2}\hat{H}} e^{-\frac{\beta}{2}\hat{H}} |\psi\rangle = \langle\psi| e^{-\beta\hat{H}} |\psi\rangle$$

$$= \underbrace{\mathrm{Tr}\,e^{-\beta\hat{H}}}_{>0} \langle\psi|\hat{\varrho}|\psi\rangle$$

$\Rightarrow \langle\psi|\hat{\varrho}|\psi\rangle \ge 0$ (2)

$\mathrm{Tr}\,\frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\,e^{-\beta\hat{H}}} = \frac{1}{\mathrm{Tr}\,e^{-\beta\hat{H}}}\mathrm{Tr}\,e^{-\beta\hat{H}} = 1$ (1)