Problem 13:

Lecture $\Rightarrow$ $$Z_N(T, B=0) = \left[e^{\beta\varepsilon}\left(1 + \sqrt{1-(1-e^{-4\beta\varepsilon})}\right)\right]^N$$

$$= \left[e^{\beta\varepsilon}(1+e^{-2\beta\varepsilon})\right]^N = \left[2\cosh\beta\varepsilon\right]^N$$

$$\Rightarrow \quad Z_N(T, B=0) = \sum_{\{S_i\}} e^{\beta\varepsilon\sum_{i=1}^{N} S_i S_{i+1}}$$

$$\Rightarrow \quad \langle S_1 S_2\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle S_i S_{i+1}\rangle = \frac{1}{\beta N}\frac{\partial}{\partial\varepsilon}\ln Z_N(T, B=0)$$

$$= \beta k_B T\,\frac{\sinh\beta\varepsilon}{\cosh\beta\varepsilon} = \tanh\beta\varepsilon \quad 4$$

$T\to 0$: $\beta\to\infty$: $\langle S_1 S_2\rangle \longrightarrow 1$ 2

$T\to\infty$: $\beta\to 0$: $\langle S_1 S_2\rangle \approx \beta\varepsilon \to 0$ 2

By the way: $\varepsilon<0 \to$ antiferromagnet

$\Rightarrow T\to 0 \Rightarrow \langle S_1 S_2\rangle \longrightarrow -1$

Problem 14:

a) $$H = -\varepsilon \sum_{\{i,j\}} S_i S_j = -\frac{\varepsilon}{2}\sum_{i=1}^{N} S_i \sum_{\substack{j \text{ neighbors} \\ \text{of } i}} S_j \approx \overbrace{-\frac{\varepsilon}{2}\sum_{i=1}^{N}\sum_{\text{j nghs}} (S_i - \langle s\rangle)(S_j - \langle s\rangle)}^{\text{neglect}}$$

$$- \underbrace{\frac{\varepsilon}{2} N \nu \langle s\rangle^2}_{\text{irrelevant constant}} + \varepsilon \nu \langle s\rangle \sum_{i=1}^{N} S_i$$

$$Z = \sum_{\{S_i\}} e^{\beta \varepsilon \nu \langle s\rangle \sum_{i=1}^{N} S_i} = \left(\sum_{s=-1}^{1} e^{\beta \varepsilon \nu \langle s\rangle s}\right)^N = \left(1 + 2\cosh \beta \varepsilon \nu \langle s\rangle\right)^N$$ (1)

$$\langle s\rangle = \frac{1}{N}\langle \sum_{i=1}^{N} S_i\rangle = \frac{1}{N\beta \nu \langle s\rangle}\left(\frac{\partial \ln Z}{\partial \varepsilon}\right)_{\beta, \nu, \langle s\rangle}$$

$$= \frac{1}{N\beta \nu \langle s\rangle} N \frac{2\sinh \beta \varepsilon \nu \langle s\rangle}{1 + 2\cosh \beta \varepsilon \nu \langle s\rangle} \beta \nu \langle s\rangle = \frac{2\sinh \beta \varepsilon \nu \langle s\rangle}{1 + 2\cosh \beta \varepsilon \nu \langle s\rangle}$$ (2)

find solutions of

$$\frac{k_B T}{\varepsilon \nu} x = \frac{2\sinh x}{1 + 2\cosh x} \equiv g(x) \quad \left(x = \frac{\varepsilon \nu}{k_B T}\langle s\rangle\right)$$

if $g'(0) > \frac{k_B T}{\varepsilon \nu}$: three solutions

if $g'(0) < \frac{k_B T}{\varepsilon \nu}$: one solution

$$\Rightarrow \frac{k_B T_c}{\varepsilon \nu} = g'(0) = \frac{2\cosh x\,[1 + 2\cosh x] - 2\sinh x \cdot 2\sinh x}{[1 + 2\cosh(x)]^2}\Bigg|_{x=0} = \frac{2}{3}$$

$$\Rightarrow T_c = \frac{2\varepsilon \nu}{3 k_B}$$ (1)

b) $T > T_c$: $x = 0 \Rightarrow \langle s\rangle = 0$ (1)

$T \to 0$: $\beta \to \infty \Rightarrow \dfrac{2\sinh \beta \varepsilon \nu \langle s\rangle}{1 + 2\cosh \beta \varepsilon \nu \langle s\rangle} \longrightarrow \pm 1$ (1)

$$\Rightarrow \langle s\rangle = \pm 1$$

$T < T_c$ but $T \approx T_c$: $\dfrac{T}{T_c} = 1 - \delta \qquad \delta \ll 1$

$$\beta \varepsilon \upsilon = \frac{\varepsilon \upsilon}{k_B T} = \frac{3}{2}\frac{2\varepsilon\upsilon}{3k_B T} = \frac{3}{2}\frac{T_c}{T}$$

$$\langle s\rangle = \frac{2\sinh \frac{3}{2}\frac{T_c}{T}\langle s\rangle}{1+2\cosh\frac{3}{2}\frac{T_c}{T}\langle s\rangle} = \frac{2\sinh \frac{3}{2}\frac{\langle s\rangle}{1-\delta}}{1+2\cosh\frac{3}{2}\frac{\langle s\rangle}{1-\delta}}$$

$$= g(0) + g'(0)\left(\frac{3}{2}\frac{\langle s\rangle}{1-\delta}\right) + \frac{1}{2}g''(0)\left(\frac{3}{2}\frac{\langle s\rangle}{1-\delta}\right)^2 + \frac{1}{6}g'''(0)\left(\frac{3}{2}\frac{\langle s\rangle}{1-\delta}\right)^3 + \cdots$$

$$= 0 + \frac{2}{3}\frac{3}{2}\frac{\langle s\rangle}{1-\delta} + 0 - \frac{1}{9}\frac{27}{8}\frac{\langle s\rangle^3}{(1-\delta)^3} + \cdots$$

$$\frac{3}{8}\frac{\langle s\rangle^3}{(1-\delta)^3} = \langle s\rangle\left(\frac{1}{1-\delta}-1\right) = \langle s\rangle\frac{\delta}{1-\delta}$$

$$\frac{3}{8}\frac{\langle s\rangle^2}{(1-\delta)^2} = \delta \qquad \langle s\rangle^2 = \frac{8}{3}\delta + O(\delta^2)$$

$$\langle s\rangle \approx \pm\sqrt{\frac{8\delta}{3}}$$

2

c) $U = \langle H\rangle = -N\frac{\varepsilon}{2}\upsilon\langle s\rangle^2$

$$C = \left(\frac{\partial U}{\partial T}\right)_N = -N\varepsilon\upsilon\langle s\rangle\left(\frac{\partial\langle s\rangle}{\partial T}\right)_N$$

$$\left(\frac{\partial\langle s\rangle}{\partial T}\right)_N = g'(\beta\varepsilon\upsilon\langle s\rangle)\left[-\frac{\varepsilon\upsilon\langle s\rangle}{k_B T^2} + \frac{\varepsilon\upsilon}{k_B T}\frac{\partial\langle s\rangle}{\partial T}\right]$$

$$\left(\frac{\partial\langle s\rangle}{\partial T}\right)_N\left[\frac{3}{2}\frac{T_c}{T}g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)-1\right] = \frac{3}{2}\frac{T_c}{T^2}\langle s\rangle g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)$$

$$\left(\frac{\partial\langle s\rangle}{\partial T}\right)_N = \frac{\frac{3}{2}\frac{T_c}{T^2}\langle s\rangle g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)}{\frac{3}{2}\frac{T_c}{T}g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)-1} = \frac{3\frac{1}{T_c}\langle s\rangle g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)}{3\frac{T}{T_c}g'\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)-2\left(\frac{T}{T_c}\right)^2}$$

$$g'(x) = \frac{4+2\cosh x}{(1+2\cosh x)^2}$$

$$C = -\frac{N\varepsilon\upsilon\langle s\rangle^2}{T_c}\,\frac{12+6\cosh\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)}{3\frac{T}{T_c}\left[4+2\cosh\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)\right]-2\left(\frac{T}{T_c}\right)^2\left[1+2\cosh\left(\frac{3}{2}\frac{T_c}{T}\langle s\rangle\right)\right]}$$

2

$T > T_c \rightarrow \langle s \rangle = 0 \Rightarrow C = 0$

$T \rightarrow 0 \rightarrow \langle s \rangle^2 = 1$, $\cosh^2$ term dominant

$$C \simeq \frac{N \varepsilon u T_c^2}{T_c T^2} \frac{6 \cosh \frac{3}{2}\frac{T_c}{T}}{8 \cosh^2 \frac{3}{2}\frac{T_c}{T}} \simeq \frac{3 N \varepsilon u T_c}{2 T^2} \frac{e^{\frac{3}{2}\frac{T_c}{T}} 4}{4 e^{2\frac{3}{2}\frac{T_c}{T}}}$$

$$\simeq \frac{3 N \varepsilon u T_c}{2 T^2} e^{-\frac{3}{2}\frac{T_c}{T}} = \frac{9}{4} N k_B \left(\frac{T_c}{T}\right)^2 e^{-\frac{3}{2}\frac{T_c}{T}} \quad 1$$

$T < T_c$ but $T \simeq T_c$ $\qquad \frac{T}{T_c} = 1 - \delta \quad 0 < \delta \ll 1 \quad \langle s \rangle = \pm\sqrt{\frac{8\delta}{3}}$

$$C \simeq -\frac{N \varepsilon u \, 8\delta}{3 T_c} \frac{18}{3(1-\delta)\left[4 + 2\cosh\left(\frac{3}{2}\sqrt{\frac{8\delta}{3}} \frac{1}{1-\delta}\right)\right] - 2(1-\delta)^2\left[1 + 2\cosh\left(\frac{3}{2}\sqrt{\frac{8\delta}{3}}\frac{1}{1-\delta}\right)\right]^2}$$

$$\simeq -4 N k_B \delta \frac{18}{3(1-\delta)\left[6 + \frac{9}{4}\frac{8\delta}{3(1-\delta)^2}\right] - 2(1-\delta)^2\left[3 + \frac{9}{4}\frac{8\delta}{3(1-\delta)^2}\right]^2}$$

$$\simeq -4 N k_B \delta \frac{18}{3(1-\delta)\, 6(1+\delta) - 2(1-2\delta)(3 + 6\delta)^2 + O(\delta^2)}$$

$$\simeq -4 N k_B \delta \frac{18}{18 - 36\delta - 18 + O(\delta^2)} = 2 N k_B + O(\delta)$$

1

Problem 15:

a) $$\mathcal{Z}(\mu',T) = \sum_{N=0}^{\infty} \frac{e^{\beta\mu' N}}{C_N} \int d\vec{x}^N \, e^{-\beta H(N,\vec{x}^N)}$$

$$= \sum_{N=0}^{\infty} \frac{e^{\beta\mu' N}}{h^{3N} N!} V^N \left(\int d\vec{p}\, e^{-\frac{\beta \vec{p}^2}{2m}} \right)^N$$ 1

$$= \sum_{N=0}^{\infty} \frac{e^{\beta\mu' N}}{h^{3N} N!} V^N \sqrt{2\pi m k_B T}^{\,3N} = \exp\left[V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'} \right]$$ 2

b) $$\Omega = -k_B T \ln \mathcal{Z}(\mu',T) = -k_B T V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'}$$ 2

c) $$P = -\left(\frac{\partial \Omega}{\partial V} \right)_{\mu',T} = +k_B T \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'}$$ 1

$$N = -\left(\frac{\partial \Omega}{\partial \mu'} \right)_{V,T} = V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'}$$ 1

$$\Rightarrow PV = k_B T \, V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'} = N k_B T$$ 1

d) $$\Omega = -k_B T \ln \sum e^{-\beta E_i + \beta\mu' N_i}$$

$$\left(\frac{\partial \Omega}{\partial \mu'} \right)_T = -\frac{\sum N_i e^{-\beta E_i + \beta\mu' N_i}}{\sum e^{-\beta E_i + \beta\mu' N_i}} = -\langle N \rangle$$

$$\left(\frac{\partial^2 \Omega}{\partial \mu'^2} \right)_T = -\frac{\beta \left(\sum N_i^2 e^{-\beta E_i + \beta\mu' N_i} \right)\left(\sum e^{-\beta E_i + \beta\mu' N_i} \right) - \beta \left(\sum N_i e^{-\beta E_i + \beta\mu' N_i} \right)^2}{\left(\sum e^{-\beta E_i + \beta\mu' N_i} \right)^2}$$

$$= -\beta \langle N^2 \rangle + \beta \langle N \rangle^2 = -\beta \left[\langle N^2 \rangle - \langle N \rangle^2 \right]$$

$$\Rightarrow \langle N^2 \rangle - \langle N \rangle^2 = -k_B T \left(\frac{\partial^2 \Omega}{\partial \mu'^2} \right)_T = +k_B T \left(\frac{\partial}{\partial \mu'} V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'} \right)_T$$ 2

$$= V \left(\frac{2\pi m k_B T}{h^2} \right)^{3/2} e^{\beta\mu'} = N$$ 1

$$\frac{\sqrt{\langle N^2 \rangle - \langle N \rangle^2}}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} \qquad \frac{1}{\sqrt{N}} = 1.3 \cdot 10^{-12} \text{ for } N = 6 \cdot 10^{23}$$ 1