Problem 7:

a) $$Z(T) = \frac{1}{C_N}\int dx^N e^{-\beta H(x^N)} = \frac{V^N}{h^{3N}N!}\int d^3\vec{p}_1 \ldots d^3\vec{p}_N\, e^{-c\beta\sum_{i=1}^{N}|\vec{p}_i|}$$
(1)

$$= \frac{V^N}{h^{3N}N!}\left(\int d^3\vec{p}\, e^{-c\beta|\vec{p}|}\right)^N$$ (2)

$$= \frac{V^N}{h^{3N}N!}\left(4\pi\int_0^\infty dp\, p^2 e^{-c\beta p}\right)^N = \frac{V^N}{h^{3N}N!}\left(\frac{4\pi k_B^3T^3}{c^3}\underbrace{\int_0^\infty dp\, p^2 e^{-p}}_{=2}\right)^N$$

polar coordinates

$$= \frac{1}{N!}\left(\frac{V 8\pi k_B^3 T^3}{c^3 h^3}\right)^N$$ (2)

b) $$F = -k_BT\ln Z(T) = -Nk_BT\ln\left[8\pi V\left(\frac{k_BT}{hc}\right)^3\right] + k_BT\overbrace{\ln N!}^{\approx N\ln N - N}$$
(1)

$$= -Nk_BT\left\{1 + \ln\left[8\pi\frac{V}{N}\left(\frac{k_BT}{hc}\right)^3\right]\right\}$$ (2)

c) $$P = -\left(\frac{\partial F}{\partial V}\right)_{N,T} = -(-Nk_BT)\left(\frac{\partial \ln V}{\partial V}\right)_{N,T} = \frac{Nk_BT}{V}$$
(1)

$$\Rightarrow PV = Nk_BT$$ (1)

d) $$U = -\left(\frac{\partial}{\partial\beta}\ln Z(T)\right)_{N,V} = -\left(\frac{\partial}{\partial\beta}N\ln\beta^{-3}\right)_{N,V} = \frac{3N}{\beta} = 3Nk_BT$$
(1) (1)

Problem 8

a) $$Z(T) = \sum_{\{n_i\}} e^{-\beta(E_0 - Lf)} = \sum_{n_1 \in \{a,b\}} \cdots \sum_{n_N \in \{a,b\}} e^{-\beta \sum_{i=1}^{N} \varepsilon_{n_i}} e^{+\beta f \sum_{i=1}^{N} l_{n_i}}$$

$$\underset{2}{=} \left(\sum_{n \in \{a,b\}} e^{-\beta \varepsilon_n + \beta f l_n} \right)^N = \left(e^{-\beta \varepsilon_a + \beta f l_a} + e^{-\beta \varepsilon_b + \beta f l_b} \right)^N \quad 1$$

b) $$U = -\left(\frac{\partial \ln Z(T)}{\partial \beta} \right) = +N \, \frac{(\varepsilon_a - f l_a) e^{-\beta \varepsilon_a + \beta f l_a} + (\varepsilon_b - f l_b) e^{-\beta \varepsilon_b + \beta f l_b}}{e^{-\beta \varepsilon_a + \beta f l_a} + e^{-\beta \varepsilon_b + \beta f l_b}} \quad 2$$

c) $$\langle L \rangle = \langle \sum_{i=1}^{N} l_{n_i} \rangle = \frac{1}{\beta} \left(\frac{\partial \ln Z(T)}{\partial f} \right)_{\beta, N} \quad 2$$

$$= N \frac{l_a e^{-\beta \varepsilon_a + \beta f l_a} + l_b e^{-\beta \varepsilon_b + \beta f l_b}}{e^{-\beta \varepsilon_a + \beta f l_a} + e^{-\beta \varepsilon_b + \beta f l_b}} \quad 2$$

at $\varepsilon_a = \varepsilon_b$ and $f = 0$:

$$\langle L \rangle = N \frac{l_a e^{-\beta \varepsilon_a} + l_b e^{-\beta \varepsilon_a}}{e^{-\beta \varepsilon_a} + e^{-\beta \varepsilon_a}} = N \frac{l_a + l_b}{2} \quad 1$$

If (roughly) half of the building blocks are in state a and half of the building blocks are in state b the entropy is maximized. In the absence of energetic reasons ($\varepsilon_a = \varepsilon_b$, $f = 0$) this is therefore the preferred state. 2

Problem 9:

$$P_{p_r}(p) = \int d\vec{x}^N \, \rho(\vec{x}^N) \, \delta(|\vec{p}_1 - \vec{p}_2| - p) \qquad 3$$

$$= \frac{V^N \int d\vec{p}_1 \cdots \int d\vec{p}_N \, e^{-\sum_{i=1}^N \frac{\beta}{2m}\vec{p}_i^{\,2}} \, \delta(|\vec{p}_1 - \vec{p}_2| - p)}{V^N \int d\vec{p}_1 \cdots \int d\vec{p}_N \, e^{-\sum_{i=1}^N \frac{\beta}{2m}\vec{p}_i^{\,2}}}$$

$$= \frac{\int d\vec{p}_1 \int d\vec{p}_2 \, e^{-\frac{\beta}{2m}\vec{p}_1^{\,2} - \frac{\beta}{2m}\vec{p}_2^{\,2}} \, \delta(|\vec{p}_1 - \vec{p}_2| - p)}{\left(\int d\vec{p} \, e^{-\frac{\beta}{2m}p^2}\right)^2} \qquad 2$$

$$= \frac{\int d\vec{P} \int d\vec{p} \, e^{-\frac{\beta}{m}\vec{P}^2} e^{-\frac{\beta}{4m}\vec{p}^{\,2}} \, \delta(|\vec{p}| - p)}{\sqrt{2\pi m k_B T}^{\,6/2}} \qquad 3$$

$$= \frac{\sqrt{m\pi k_B T}^{\,3}}{(2\pi m k_B T)^3} \int d\vec{p} \, e^{-\frac{\beta}{4m}\vec{p}^{\,2}} \, \delta(|\vec{p}| - p)$$

$$= \frac{1}{8(m\pi k_B T)^{3/2}} \, 4\pi \, e^{-\frac{\beta}{4m}p^2} p^2 = \frac{1}{2\sqrt{\pi}} \frac{1}{(m k_B T)^{3/2}} p^2 e^{-\frac{\beta}{4m}p^2} \qquad 2$$