Problem 4:

a) The only two possible values are the energies $-\varepsilon_0$ and ε_0 themselves. (2)

b) The probability for the two states in the canonical ensemble which is applicable to this situation are

$$p_1 = \frac{e^{\frac{\varepsilon_0}{k_B T}}}{e^{\frac{\varepsilon_0}{k_B T}} + e^{-\frac{\varepsilon_0}{k_B T}}} \qquad p_2 = \frac{e^{-\frac{\varepsilon_0}{k_B T}}}{e^{\frac{\varepsilon_0}{k_B T}} + e^{-\frac{\varepsilon_0}{k_B T}}}$$

(2)

c)

$$U \underset{(1)}{=} p_1(-\varepsilon_0) + p_2(+\varepsilon_0) = -\varepsilon_0 \frac{e^{\frac{\varepsilon_0}{k_B T}} - e^{-\frac{\varepsilon_0}{k_B T}}}{e^{\frac{\varepsilon_0}{k_B T}} + e^{-\frac{\varepsilon_0}{k_B T}}}$$

$$= -\varepsilon_0 \tanh \frac{\varepsilon_0}{k_B T} \qquad (1)$$

d) $T \to 0 \Rightarrow \frac{\varepsilon_0}{k_B T} \to +\infty \Rightarrow \tanh \frac{\varepsilon_0}{k_B T} \to 1 \Rightarrow U \to -\varepsilon_0$

$T \to \infty \Rightarrow \frac{\varepsilon_0}{k_B T} \to 0 \Rightarrow U \to 0$ (1)

all values in between are possible, i.e., $]-\varepsilon_0, 0[$ (1)

(if $T \to 0$ <u>from below</u> $U \to \varepsilon_0$, i.e., if negative temperatures are possible (depends on the heat bath), the range is $[-\varepsilon_0, \varepsilon_0]$)

Problem 5.

a) $$Z(T) \underset{2}{=} \sum_{n_1\in\{L,R,S\}} \cdots \sum_{n_N\in\{L,R,S\}} e^{-\beta\sum_{i=1}^{N}\left\{\begin{matrix}\varepsilon & n_i\in\{L,R\}\\ 0 & n_i=S\end{matrix}\right\}}$$

$$\underset{1}{=} \left(\sum_{n\in\{L,R,S\}} e^{-\beta\left\{\begin{matrix}\varepsilon & n\in\{L,R\}\\ 0 & n=S\end{matrix}\right\}}\right)^N$$

$$= (1+2e^{-\beta\varepsilon})^N \quad 2$$

b) $$U \underset{1}{=} -\left(\frac{\partial \ln Z}{\partial\beta}\right)_N = -N\left(\frac{\partial \ln(1+2e^{-\beta\varepsilon})}{\partial\beta}\right)_N = -N\,\frac{2e^{-\beta\varepsilon}(-\varepsilon)}{1+2e^{-\beta\varepsilon}}$$

$$= \frac{2N\varepsilon}{e^{\beta\varepsilon}+2} \quad 2$$

c) They are identical. 2

If they are not identical because of some mistake in parts a) or b) accept only if it is acknowledged that something must be wrong.

Problem 6:

a) $$Z(T) = \frac{1}{C_N}\int d\vec{x}^{N}\, e^{-\beta H(\vec{x}^N)}$$

$$= \frac{1}{h^{3N}N!} A^N \int_0^\infty dz_1 \ldots \int_0^\infty dz_N \int d^3\vec{p}_1 \ldots \int d^3\vec{p}_N\, e^{-\frac{\beta}{2m}\sum_{i=1}^N \vec{p}_i^{\,2}}\, e^{-\beta mg\sum_{i=1}^N z_i} \quad 2$$

$$= \frac{1}{h^{3N}N!} A^N \left(\int_0^\infty dz\, e^{-\beta mgz}\right)^N \left(\int d^3\vec{p}\, e^{-\frac{\beta}{2m}\vec{p}^{\,2}}\right)^N$$

$$= \frac{1}{N!}\left[\frac{A}{h^3}\,\frac{k_BT}{mg}\,(2\pi m k_BT)^{3/2}\right]^N \quad 2$$

b) $$\rho(\vec{r}) = \int d\vec{x}^N \frac{e^{-\beta H(\vec{x}^N)}}{C_N Z(T)}\,\delta(\vec{r}_1-\vec{r}) \quad 2$$

$$= \left(\frac{mg}{Ak_BT}\right)^N (2\pi m k_BT)^{-\frac{3N}{2}} \int d^3\vec{r}_1 \ldots \int d^3\vec{r}_N \int d^3\vec{p}_1 \ldots \int d^3\vec{p}_N\, e^{-\frac{\beta}{2m}\sum_{i=1}^N \vec{p}_i^{\,2}}\, e^{-\beta mg\sum_{i=1}^N z_i}\,\delta(\vec{r}_1-\vec{r})$$

$$= \left(\frac{mg}{Ak_BT}\right)^N (2\pi m k_BT)^{-\frac{3N}{2}} \left(\int d^3\vec{p}\, e^{-\frac{\beta}{2m}\vec{p}^{\,2}}\right)^N A^{N-1}\left(\int_0^\infty dz\, e^{-\beta mgz}\right)^{N-1} \int d^3\vec{r}_1\, e^{-\beta mgz_1}\,\delta(\vec{r}_1-\vec{r})$$

$$= \left(\frac{mg}{Ak_BT}\right)^N (2\pi m k_BT)^{-\frac{3N}{2}} (2\pi m k_BT)^{\frac{3N}{2}} A^{N-1}\left(\frac{k_BT}{mg}\right)^{N-1} e^{-\beta mgz}$$

$$= \frac{mg}{Ak_BT}\, e^{-\beta mgz} \quad 2$$

c) law of large numbers →
there are $N\rho(\vec{r})A\Delta z$ particles within the volume $A\Delta z$ 1

$$\Rightarrow P(z)\,\overbrace{A\Delta z}^{V} = \overbrace{N\rho(\vec{r})\,A\Delta z}^{\text{"}N\text{"}}\, k_BT$$

$$\Rightarrow P(z) = \rho(\vec{r})Nk_BT = \frac{Nmg}{A}\, e^{-\frac{mgz}{k_BT}} = P(0)\, e^{-\frac{mgz}{k_BT}} \quad 1$$