$$C_N = \frac{N k_B \langle s\rangle^2}{\left(\frac{T}{T_c}\right)^2 \cosh^2\left(\frac{T_c}{T}\langle s\rangle\right) - \frac{T}{T_c}}$$

$T > T_c$: $\quad \langle s\rangle = 0 \Rightarrow C_N = 0$

$T \ll T_c$: $\quad \langle s\rangle = \pm 1 \Rightarrow C_N = \dfrac{N k_B}{\underbrace{\left(\frac{T}{T_c}\right)^2 \cosh^2\left(\frac{T_c}{T}\right)}_{\approx \left(\frac{e^{T_c/T}}{2}\right)^2} - \frac{T}{T_c}} \sim 4 N k_B \left(\frac{T_c}{T}\right)^2 e^{-2\frac{T_c}{T}}$

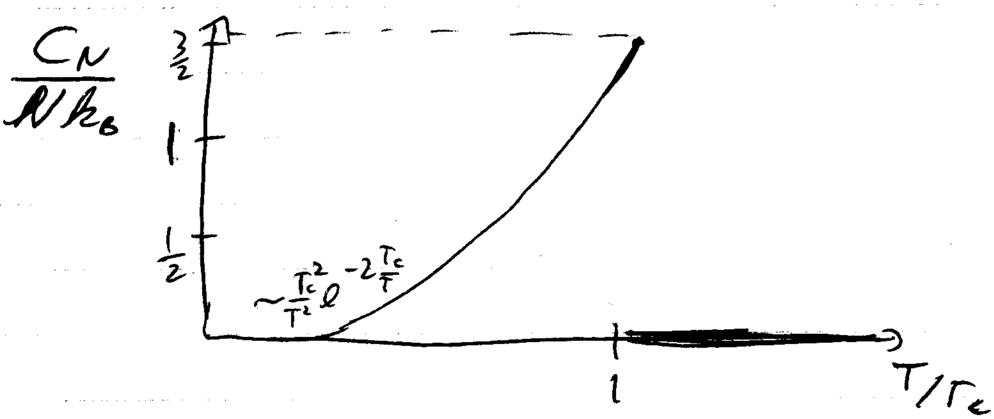
small compared to

$T < T_c$ but close to T_c, $\quad \frac{T}{T_c} = 1 - \delta$, $\quad \langle s\rangle \approx \sqrt{3\delta}$

$$C_N \approx \frac{N k_B \cdot 3\delta}{(1-\delta)^2 \cosh^2\left(\frac{\sqrt{3\delta}}{1-\delta}\right) - (1-\delta)} \approx \frac{3 N k_B \delta}{(1-\delta)^2\left(1 + \frac{1}{2}\frac{3\delta}{(1-\delta)^2}\right)^2 - 1 + \delta}$$

$$\approx \frac{3 N k_B \delta}{-2\delta + 2\frac{3}{2}\delta + \delta} = \frac{3}{2} N k_B$$

$\rightarrow$ jump in C_N $\rightarrow$ second order transition



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$\downarrow$ 2/3

Magnetic susceptibility:

$$\chi_{T,N}(B) = \left(\frac{\partial \langle M\rangle}{\partial B}\right)_{T,N} = N\mu \left(\frac{\partial \langle s\rangle}{\partial B}\right)_{T,N}$$

$$\langle s\rangle = \tanh\left[\frac{T_c}{T}\langle s\rangle + \beta\mu B\right]$$

$$\left(\frac{\partial\langle s\rangle}{\partial B}\right)_{T,N} = \frac{\frac{T_c}{T}\left(\frac{\partial\langle s\rangle}{\partial B}\right)_{T,N} + \beta\mu}{\cosh^2\left[\frac{T_c}{T}\langle s\rangle + \beta\mu B\right]}$$

$$\Rightarrow \left(\frac{\partial\langle s\rangle}{\partial B}\right)_{T,N} = \frac{\beta\mu}{\cosh^2\left[\frac{T_c}{T}\langle s\rangle + \beta\mu B\right] - \frac{T_c}{T}}$$

at $B=0$:

$$\chi_{T,N}(B) = N\mu \frac{\beta\mu}{\cosh^2\left[\frac{T_c}{T}\langle s\rangle\right] - \frac{T_c}{T}}$$

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$T > T_c$: $\langle s\rangle = 0$ $\qquad \chi_{T,N}(B) = \dfrac{N\mu^2\beta}{1-\frac{T_c}{T}} = \dfrac{N\mu^2}{k_B}\dfrac{1}{T-T_c}$

$T \ll T_c$: $\qquad \chi_{T,N}(B) = \dfrac{N\mu^2}{k_B}\dfrac{1}{T\underbrace{\cosh^2\left(\frac{T_c}{T}\right)}_{\simeq (2e^{T_c/T})^2} - T_c} \simeq \dfrac{4N\mu^2}{k_B T} e^{-2\frac{T_c}{T}}$

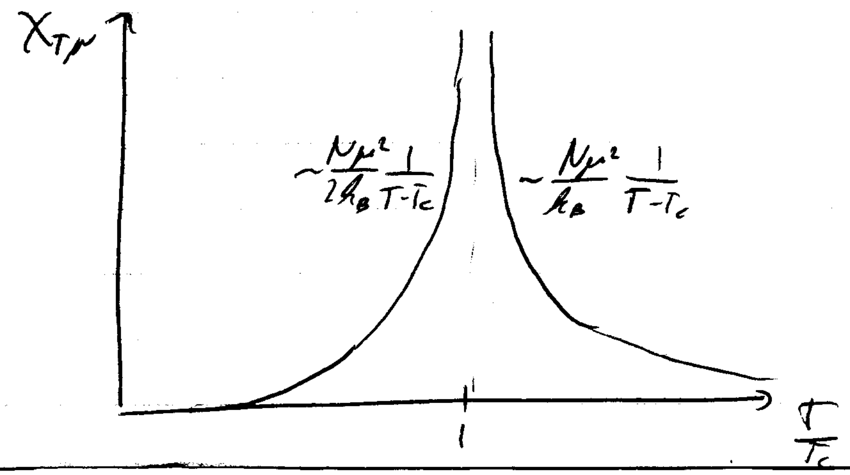
$T < T_c$ but $T \simeq T_c$ $\qquad \dfrac{T}{T_c} = 1-\delta \qquad 0 < \delta \ll 1$

$\langle s\rangle \approx \sqrt{3\delta}$

$$\chi_{T,N}(B) \simeq \frac{N\mu^2}{k_B T_c}\frac{1}{1-\delta}\frac{1}{\cosh^2\left(\frac{\sqrt{3\delta}}{1-\delta}\right) - \frac{1}{1-\delta}}$$

$$\simeq \frac{N\mu^2}{k_B T_c}\frac{1}{1-\delta}\frac{1}{\left(1+\frac{1}{2}\frac{3\delta}{(1-\delta)^2}\right)^2 - 1 - \delta} \simeq \frac{N\mu^2}{k_B T_c}\frac{1}{2\delta} = \frac{N\mu^2}{2k_B}\frac{1}{T-T_c}$$

$\chi_{T,N}(B)$ diverges like $\dfrac{1}{T-T_c}$ close to T_c



V.4 Grand Canonical Ensemble

V.4.1 General framework (discrete system)

Physical situation: open system in contact with heat bath and particle bath

Describe system by $\mathcal{N}$ states $i = 1, \ldots, \mathcal{N}$ with
- energy of state i: E_i
- number of particles in state i: N_i

System characterized by probabilities p_i of state i.
- heat bath fixes average energy $U = \langle E \rangle = \sum_{i=1}^{\mathcal{N}} p_i E_i$
- particle bath fixes average number of particles $N = \sum_{i=1}^{\mathcal{N}} p_i N_i$
- maximize entropy $S = -k_B \sum_{i=1}^{\mathcal{N}} p_i \ln p_i$ under these conditions and $\sum_{i=1}^{\mathcal{N}} p_i = 1$

→ Lagrange multipliers λ, μ and ν

$$0 = \frac{\partial}{\partial p_i}\left[-\sum_{i=1}^{\mathcal{N}} p_i \ln p_i + \lambda\left(\sum_{i=1}^{\mathcal{N}} p_i - 1\right) + \mu\left(U - \sum_{i=1}^{\mathcal{N}} p_i E_i\right) + \nu\left(N - \sum_{i=1}^{\mathcal{N}} p_i N_i\right)\right]$$

$$= -\ln p_i - 1 + \lambda - \mu E_i - \nu N_i$$

$$\Rightarrow p_i = e^{\lambda - 1}\, e^{-\mu E_i}\, e^{-\nu N_i}$$

still have to find λ, μ, and ν

λ from normalization

$$p_i = \frac{e^{-\mu E_i - \nu N_i}}{\sum_{j=1}^{\mathcal{N}} e^{-\mu E_j - \nu N_j}}$$

μ and ν?

$$S = -k_B \sum_{i=1}^{N} p_i \ln p_i = -\frac{k_B}{\sum_{j=1}^{N} e^{-\mu E_j - \nu N_j}} \sum_{i=1}^{N} e^{-\mu E_i - \nu N_i} \ln \frac{e^{-\mu E_i - \nu N_i}}{\sum_{j=1}^{N} e^{-\mu E_j - \nu N_j}}$$

$$= k_B \ln \sum_{j=1}^{N} e^{-\mu E_j - \nu N_j} + k_B \sum_{i=1}^{N} (\mu E_i + \nu N_i) \frac{e^{-\mu E_i - \nu N_i}}{\sum_{j=1}^{N} e^{-\mu E_j - \nu N_j}}$$

$$= k_B \ln \sum_{j=1}^{N} e^{-\mu E_j - \nu N_j} + k_B \mu U + k_B \nu N$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_N = k_B \mu + k_B U \left(\frac{\partial \mu}{\partial U}\right)_N + k_B \left(\frac{\partial \nu}{\partial U}\right)_N N$$
$$+ k_B \frac{\sum_{i=1}^{N} \left[-E_j \left(\frac{\partial \mu}{\partial U}\right)_N - N_j \left(\frac{\partial \nu}{\partial U}\right)_N\right] e^{-\mu E_j - \nu N_j}}{\sum_{j=1}^{N} e^{-\mu E_j - \nu N_j}}$$

$$= k_B \mu + k_B U \left(\frac{\partial \mu}{\partial U}\right)_N + k_B \left(\frac{\partial \nu}{\partial U}\right)_N N - k_B U \left(\frac{\partial \mu}{\partial U}\right)_N - k_B N \left(\frac{\partial \nu}{\partial U}\right)_N$$
$$= k_B \mu$$

$$\Rightarrow \boxed{\mu = \frac{1}{k_B T} = \beta}$$

$$\frac{\mu'}{T} = -\left(\frac{\partial S}{\partial N}\right)_U = -k_B \nu - k_B N \left(\frac{\partial \nu}{\partial N}\right)_U - k_B U \left(\frac{\partial \mu}{\partial N}\right)_U$$
$$- k_B \frac{\sum_{i=1}^{N} \left[-E_i \left(\frac{\partial \mu}{\partial N}\right)_U - N_i \left(\frac{\partial \nu}{\partial N}\right)_U\right] e^{-\mu E_i - \nu N_i}}{\sum_{j=1}^{N} e^{-\mu E_j - \nu N_j}}$$

$$= -k_B \nu - k_B N \left(\frac{\partial \nu}{\partial N}\right)_U - k_B U \left(\frac{\partial \mu}{\partial N}\right)_U + k_B U \left(\frac{\partial \mu}{\partial N}\right)_U + k_B N \left(\frac{\partial \nu}{\partial N}\right)_U$$
$$= -k_B \nu$$

$$\Rightarrow \nu = -\frac{\mu'}{k_B T} = -\beta \mu'$$

Result:

$$\boxed{p_i = \frac{e^{-\beta E_i + \beta \mu' N_i}}{\sum_{j=1}^{N} e^{-\beta E_j + \beta \mu' N_j}}}$$

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