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Extra assumption:

The spins are oriented in z direction by crystal.

$$\Rightarrow H = -\varepsilon \sum_{\{i,j\}} s_i s_j - \mu B \sum_{i=1}^{N} s_i$$

with $s_i \in \{\pm 1\}$ → everything commutes

"Ising model"

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Reminder:

want to explain

at $B=0$

0 ⊢ ferromagnet $M \neq 0$ • T^* paramagnet $M = 0$ → T

$H = -\varepsilon \sum_{\{i,j\}} S_i S_j$ with $\varepsilon > 0$ promising:

- at low T energy dominates, energy wants either all spins $+1$ or all spins -1
  ⇒ ferromagnet
- at ~~high~~ T entropy dominates, entropy wants spins to be "random"
  ⇒ $M=0$, paramagnet

Results:

$d=1$: – Ising model can be solved exactly
– no ferromagnetism

$d=2$: – Ising model can be solved exactly
– very technical, Onsager 1944
– has phase transition as expected
– ~~one~~ of the few exactly solvable models in the
with a phase transition

$d=3$: – no exact solution

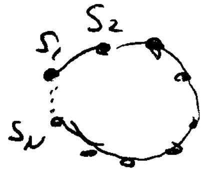
large d: – approximate solution "mean field theory"
– phase transition

V.3.2 Ising model in $d=1$

model:

N sites on a line, 2^N states

$$E = -\varepsilon \sum_{i=1}^{N} s_i s_{i+1} - \mu B \sum_{i=1}^{N} s_i$$



periodic boundary conditions: $s_1 = s_{N+1}$

partition function:

$$Z_N(T,B) = \sum_{\{S_i\}} e^{-\beta E} = \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} e^{\beta\varepsilon \sum_{i=1}^{N} s_i s_{i+1} + \beta\mu B \sum_{i=1}^{N} s_i}$$

$$= \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} e^{\beta\varepsilon s_1 s_2 + \beta\mu B s_1} \sum_{s_3=\pm 1} e^{\beta\varepsilon s_2 s_3 + \beta\mu B s_2} \cdots \sum_{s_N=\pm 1} e^{\beta s_{N-1} s_N + \beta\mu B s_{N-1}} \cdot e^{\beta s_N s_1 + \beta\mu B s_N}$$

Define "transfer matrix"

$$T_{s,s'} = e^{\beta\varepsilon s s' + \beta\mu B s}$$

$$T = \begin{array}{cc} \begin{array}{cc} -1 & \quad s' \quad 1 \end{array} & \\ \begin{pmatrix} e^{\beta\varepsilon - \beta\mu B} & e^{-\beta\varepsilon - \beta\mu B} \\ e^{-\beta\varepsilon + \beta\mu B} & e^{\beta\varepsilon + \beta\mu B} \end{pmatrix} & \begin{array}{c} -1 \\ s \\ 1 \end{array} \end{array}$$

$$Z_N(T,B) = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} T_{s_1 s_2} \sum_{s_3=\pm 1} T_{s_2 s_3} \cdots \sum_{s_N=\pm 1} T_{s_{N-1} s_N} T_{s_N s_1}$$

$$= \sum_{s_1=\pm 1} \sum_{s_3=\pm 1} (T^2)_{s_1 s_3} \sum_{s_4=\pm 1} T_{s_3 s_4} \cdots \sum_{s_N=\pm 1} T_{s_{N-1} s_N} T_{s_N s_1}$$

$$= \sum_{s_1=\pm 1} (T^N)_{s_1 s_1} = \mathrm{Tr}\, T^N = \lambda_+^N + \lambda_-^N$$

if λ_+ and λ_- are the eigenvalues of T

$$= \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right) \simeq \lambda_+^N \quad \text{for large } N$$

largest eigenvalue of T

Find largest eigenvalue of $\bar{T}$:

$$\det(\bar{T}-\lambda I) = (e^{\beta\varepsilon-\beta\mu B}-\lambda)(e^{\beta\varepsilon+\beta\mu B}-\lambda) - e^{-2\beta\varepsilon}$$

$$= \lambda^2 - \lambda e^{\beta\varepsilon}(e^{-\beta\mu B}+e^{\beta\mu B}) + e^{2\beta\varepsilon} - e^{-2\beta\varepsilon}$$

$$= \lambda^2 - 2\lambda e^{\beta\varepsilon}\cosh\beta\mu B + e^{2\beta\varepsilon} - e^{-2\beta\varepsilon}$$

$\Rightarrow$

$$\lambda_\pm = e^{\beta\varepsilon}\cosh\beta\mu B \pm \sqrt{e^{2\beta\varepsilon}\cosh^2\beta\mu B - (e^{2\beta\varepsilon}-e^{-2\beta\varepsilon})}$$

$$= e^{\beta\varepsilon}\left[\cosh\beta\mu B \pm \sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}\right]$$

Gibbs free energy (note: B fixed, not M)

$$G_N(T,B) = -k_B T \ln Z_N(T,B)$$

$$\approx -k_B T N \ln \lambda_+$$

$$= -\varepsilon N - N k_B T \ln\left[\cosh\beta\mu B + \sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}\right]$$

$$M = -\left(\frac{\partial G_N(T,B)}{\partial B}\right)_{N,T} = N k_B T \beta\mu \frac{\sinh\beta\mu B + \frac{2\cosh\beta\mu B \sinh\beta\mu B}{2\sqrt{\cosh^2\beta\mu B-(1-e^{-4\beta\varepsilon})}}}{\cosh\beta\mu B + \sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}}$$

$$= N\mu \frac{\sinh\beta\mu B \, \frac{\cosh\beta\mu B + \sqrt{\cosh^2\beta\mu B-(1-e^{-4\beta\varepsilon})}}{\sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}}}{\cosh\beta\mu B + \sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}}$$

$$= N\mu \frac{\sinh\beta\mu B}{\sqrt{\cosh^2\beta\mu B - (1-e^{-4\beta\varepsilon})}}$$

$M \to 0$ for $B \to 0$ $\Rightarrow$ no ferromagnetism at any temperature

Mermin-Wagner theorem: One-dimensional classical systems with short-range interactions do not have phase transitions

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