as $T \to 0 \quad s \sim \lambda_T^{-3} \sim T^{3/2} \to 0$

$\to$ consistent with third law of thermodynamics

$\downarrow$ 2/26

$\downarrow$ 3/2

__heat capacity per volume__

$$c_n = T\left(\frac{\partial s}{\partial T}\right)_n$$

in the condensate ($z=1$)

$$\boxed{c_n = T k_B \frac{5}{2} g_{5/2}(1)\left(\frac{\partial}{\partial T}\overbrace{\lambda_T^{-3}}^{T^{+\frac{3}{2}}}\right)_n = \frac{15}{4} k_B \frac{g_{5/2}(1)}{\lambda_T^3}}$$

in the normal phase

$$c_n = T k_B \frac{5}{2} g_{5/2}(z)\left(\frac{\partial}{\partial T}\lambda_T^{-3}\right)_n + T k_B \frac{5}{2}\frac{1}{\lambda_T^3}\left(\frac{\partial}{\partial T} g_{5/2}(z)\right)_n$$

$$- k_B n \left(\frac{\partial}{\partial T}\ln z\right)_n$$

$$= \frac{15}{4} k_B \frac{g_{5/2}(1)}{\lambda_T^3} + k_B \frac{5}{2}\underbrace{\frac{1}{\lambda_T^3}\frac{g_{3/2}(z)}{z}}_{n} T\left(\frac{\partial z}{\partial T}\right)_n - k_B n \frac{T}{z}\left(\frac{\partial z}{\partial T}\right)_n$$

$$\frac{\partial}{\partial T}\frac{g_{1/2}(z)}{\lambda_T^3}$$

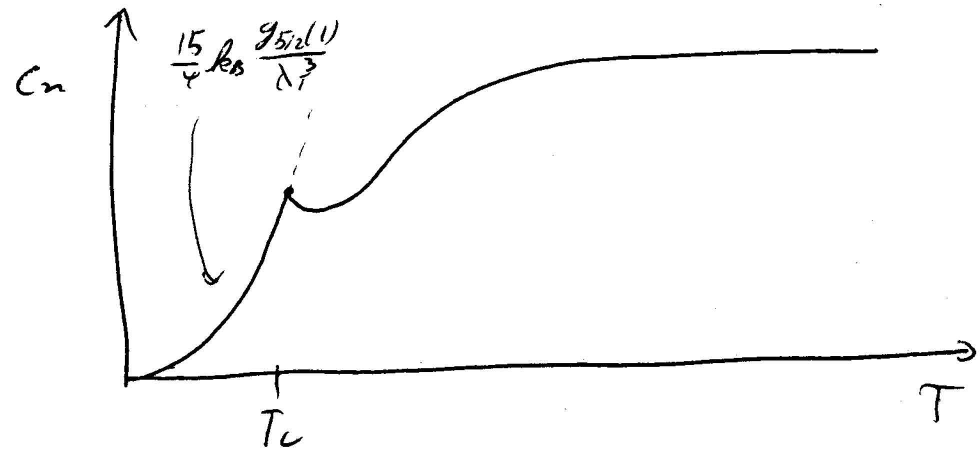
$\cancel{\frac{T}{z}\left(\frac{\partial z}{\partial T}\right)_n}$ z given by $n = \frac{1}{\lambda_T^3} g_{3/2}(z)$

$$\Rightarrow \frac{g_{1/2}(z)}{z}\left(\frac{\partial z}{\partial T}\right)_n = n \frac{\partial \lambda_T^3}{\partial T} = -\frac{3}{2}\frac{n}{T}\lambda_T^3$$

$$\Rightarrow \frac{T}{z}\left(\frac{\partial z}{\partial T}\right)_n = -\frac{3}{2}\frac{n\lambda_T^3}{g_{1/2}(z)} = -\frac{3}{2}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$$\boxed{c_n = \frac{15}{4} k_B \frac{g_{5/2}(z)}{\lambda_T^3} - \underbrace{\frac{9}{4} k_B n \frac{g_{3/2}(z)}{g_{1/2}(z)}}_{\xrightarrow[z\to 1]{} 0}}$$

$\xrightarrow[z \to 1]{} 0$ since $g_{1/2}(z)$ diverges



High temperatures:

z given by $\quad n = \frac{1}{\lambda_T^3} g_{3/2}(z) = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(z)$

$$g_{3/2}(z) = n\left(\frac{2\pi\hbar^2}{m k_B T}\right)^{3/2} \longrightarrow 0 \Rightarrow z \to 0$$

$$g \Rightarrow g_{3/2}(z) \approx z$$

$$\Rightarrow z \simeq n\left(\frac{2\pi\hbar^2}{m k_B T}\right)^{3/2} = n\lambda_T^3$$

$$\Rightarrow P \approx \frac{k_B T}{\lambda_T^3} g_{5/2}(n\lambda_T^3) \simeq \frac{k_B T}{\lambda_T^3} n\lambda_T^3 = \frac{N}{V} k_B T$$

$$\Rightarrow PV = N k_B T \quad \text{ideal gas at high temperatures}$$

$$c_n = \frac{15}{4} k_B \frac{g_{5/2}(z)}{\lambda_T^3} - \frac{9}{4} k_B n \frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$$\approx \frac{15}{4} k_B \frac{n\lambda_T^3}{\lambda_T^3} - \frac{9}{4} k_B n = \frac{3}{2} k_B n \rightarrow \text{ideal gas}$$

VI.5 The ideal Fermi-Dirac gas

$$\hat{H} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} \qquad \text{box of volume } V = L^3$$

periodic boundary conditions

$\Rightarrow$ allowed $\vec{k} = \left(\frac{2\pi}{L}\right)(l_x \hat{x} + l_y \hat{y} + l_z \hat{z}) \qquad l_x, l_y, l_z = \ldots, -1, 0, 1, \ldots$

single particle energies $\varepsilon_{\vec{k}} = \frac{\hbar^2 \vec{k}^2}{2m}$

spin $s \Rightarrow g = 2s+1$ states for each $\vec{k}$

grand canonical partition function

$$\mathcal{Z}(\mu', T) = \operatorname{Tr} e^{-\beta \hat{H} + \beta \mu' \hat{N}}$$

$$= \sum_{\{n_{\vec{k},i}\}} \prod_{\vec{k},i} e^{-\beta \varepsilon_{\vec{k}} n_{\vec{k}} + \beta \mu' n_{\vec{k}}}$$

i: spin degrees of freedom $1, \ldots, g$

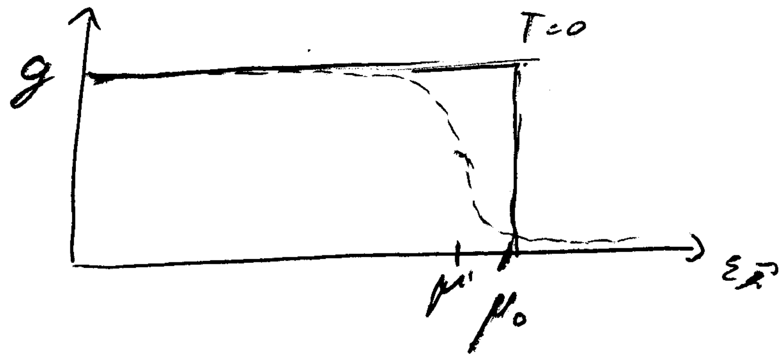
$$= \prod_{\vec{k}} \prod_{i=1}^{g} \sum_{n=0}^{1} e^{-\beta(\varepsilon_{\vec{k}} - \mu')n} = \prod_{\vec{k}} \left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \mu')}\right)^g$$

$$\Omega = -k_B T \ln \mathcal{Z}(\mu', T) = -k_B T g \sum_{\vec{k}} \ln\left(1 + e^{-\beta(\varepsilon_{\vec{k}} - \mu')}\right)$$

$$\langle N \rangle = -\left(\frac{\partial \Omega}{\partial \mu'}\right)_{T,V} = g \sum_{\vec{k}} \frac{e^{-\beta(\varepsilon_{\vec{k}} - \mu')}}{1 + e^{-\beta(\varepsilon_{\vec{k}} - \mu')}} = \sum_{\vec{k}} \frac{g z}{e^{\beta \varepsilon_{\vec{k}}} + z}$$

$$\langle n_{\vec{k}} \rangle = k_B T \left(\frac{\partial \ln \mathcal{Z}}{\partial \varepsilon_{\vec{k}}}\right)_{T,V} = \frac{g z}{e^{\beta \varepsilon_{\vec{k}}} + z} \qquad \text{finite for any } 0 \le z < \infty$$

- Plot expected number of particles at momentum $\vec{k}$ $\langle n_{\vec{k}} \rangle$ as a function of energy $\varepsilon_{\vec{k}}$



at $T=0$: chemical potential μ_0
all states with $\varepsilon_{\vec{k}} < \mu_0$ have g particles
all states with $\varepsilon_{\vec{k}} > \mu_0$ have 0 particles

$\Rightarrow$ μ_0 given by $Ng = \#$ states with energy $\leq \mu_0$

μ_0 is called Fermi energy ε_F

$\rightarrow$ Fermi momentum $p_F = \sqrt{2m\varepsilon_F}$
Fermi temperature $T_F = \frac{\varepsilon_F}{k_B}$

at $T>0$ but still low:
all states well below μ' have g particles
all states well above μ' have 0 particles
$\rightarrow$ "Fermi sea"
only states around μ' contribute to thermodynamical properties

— back to the grand potential:

$$\Omega = -k_B T g \sum_{\vec{p}} \ln\left(1 + e^{-\beta(\varepsilon_{\vec{p}} - \mu')}\right)$$

$$= -k_B T g \sum_{\vec{l}} \ln\left(1 + e^{-\beta\left(\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \vec{l}^{\,2} - \mu'\right)}\right)$$

$$\underset{L\to\infty}{=} -k_B T g \int d^3\vec{l}\, \ln\left(1 + e^{-\beta\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \vec{l}^{\,2}} z\right)$$

$$= -4\pi k_B T g \int_0^\infty dl\, l^2 \ln\left(1 + e^{-\beta\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 l^2} z\right)$$

$$x = \sqrt{\beta \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2}\, l$$

$$= -4\pi k_B T g \left(\frac{2m k_B T \pi}{h^2}\right)^{3/2} \frac{V}{\pi^{3/2}} \int_0^\infty dx\, x^2 \ln\left(1 + e^{-x^2} z\right)$$

$$= -\frac{k_B T g V}{\lambda_T^3} \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} e^{-x^2 n} z^n$$

$$= -\frac{k_B T g V}{\lambda_T^3} \frac{4}{\sqrt{\pi}} \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn}\right) \int_0^\infty e^{-nx^2} dx$$

$$= -\frac{k_B T g V}{\lambda_T^3} \frac{4}{\sqrt{\pi}} \sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn}\right) \sqrt{\frac{\pi}{n}}\, \frac{1}{2}$$

$$= -\frac{k_B T g V}{\lambda_T^3} \underbrace{\sum_{n=1}^\infty (-1)^{n+1} \frac{z^n}{n^{5/2}}}_{\equiv f_{5/2}(z)}$$

well defined for all $0 \le z <$

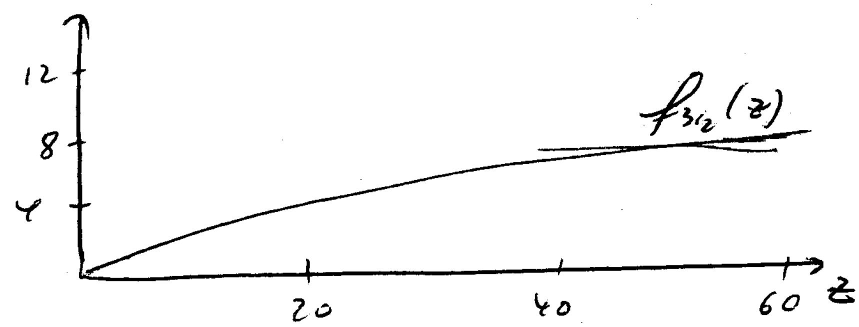
~~$\Rightarrow$ pressure $P = -\frac{\Omega}{V} = \frac{k_B T g}{\lambda_T^3}$~~

$$\langle N\rangle = -\left(\frac{\partial\Omega}{\partial\mu'}\right)_{T,V} = -\left(\frac{\partial\Omega}{\partial z}\right)_{T,V}\left(\frac{\partial z}{\partial\mu'}\right)_{T,V} = -\frac{z}{k_BT}\left(\frac{\partial\Omega}{\partial z}\right)_{T,V}$$

$$= \frac{gV}{\lambda_T^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{3/2}} = \frac{gV}{\lambda_T^3} f_{3/2}(z)$$

$$\langle n\rangle = \frac{\langle N\rangle}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z)$$

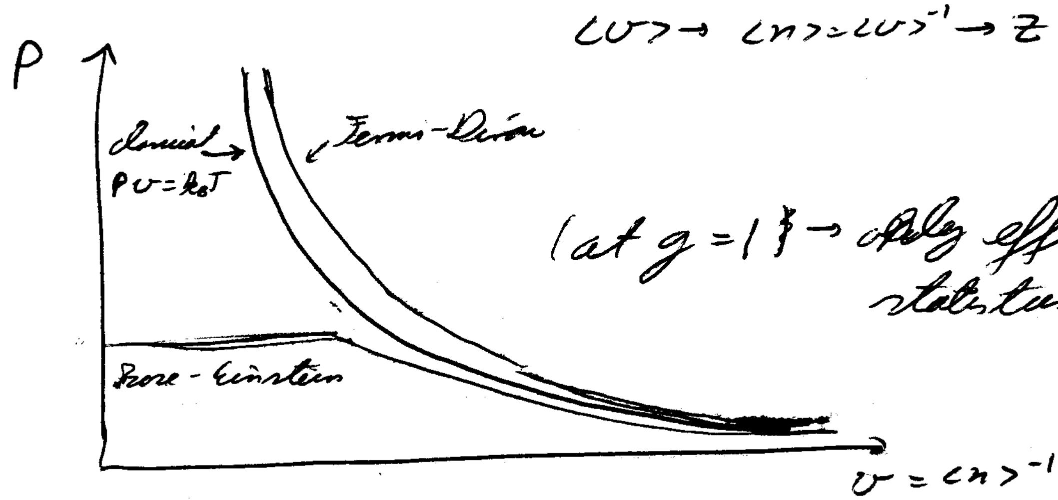
given experimental particle density $\langle n\rangle$
→ calculate fugacity z



pressure:

$$P = -\frac{\Omega}{V} = \frac{k_BTg}{\lambda_T^3} f_{5/2}(z)$$

$\langle v\rangle \to \langle n\rangle = \langle v\rangle^{-1} \to z \to P$



(at $g = 1$) → only effect of statistics)

- pressure of Bose-Einstein gas lower than classical pressure since particles in ground state do not contribute to pressure
- pressure of Fermi-Dirac gas higher than classical pressure since no state can have more than one particle → low momentum states have less particles than classically → high momentum states have more particles than classically 3/12

Low temperature thermodynamics

Given N and V what is μ'?

$$N = \int_{-\infty}^{\infty} n(\varepsilon) \frac{g}{e^{\beta(\varepsilon-\mu')}+1} \, d\varepsilon$$

$n(\varepsilon)$: density of states

$\Omega(\varepsilon)$ = # of states below energy ε

$$n(\varepsilon) = \frac{d}{d\varepsilon} \Omega(\varepsilon)$$

$\Omega(\varepsilon)$ = # of $\vec{l}$ with $\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \vec{l}^{\,2} \le \varepsilon$

$$\Rightarrow \Omega(\varepsilon) = \frac{4}{3}\pi \left(\frac{2m\varepsilon}{h^2}\right)^{3/2} V$$

$$\Rightarrow n(\varepsilon) = \begin{cases} 2\pi V \left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2} & \varepsilon > 0 \\ 0 & \varepsilon < 0 \end{cases}$$

at $T=0$: $\mu' = \varepsilon_F$ $\quad \frac{g}{e^{\beta(\varepsilon-\mu')}+1} = \begin{cases} g & \varepsilon < \varepsilon_F \\ 0 & \varepsilon > \varepsilon_F \end{cases}$