$$\langle N \rangle = -\left(\frac{\partial \Omega}{\partial \mu'}\right)_{T,V} = -\left(\frac{\partial \Omega}{\partial z}\right)_{T,V}\left(\frac{\partial z}{\partial \mu'}\right)_{T,V} = -\beta z \left(\frac{\partial \Omega}{\partial z}\right)_{T,V}$$

$$= \frac{z}{1-z} + \frac{4V}{\lambda_T^3 \sqrt{\pi}} \int_{\lambda_T \frac{\sqrt{\pi}}{L}}^{\infty} dx\, x^2 \frac{z}{e^{x^2}-z}$$

for large systems: $V \to \infty$, $\langle N \rangle \to \infty$ with

$$\langle n \rangle = \frac{\langle N \rangle}{V} \text{ fixed:}$$

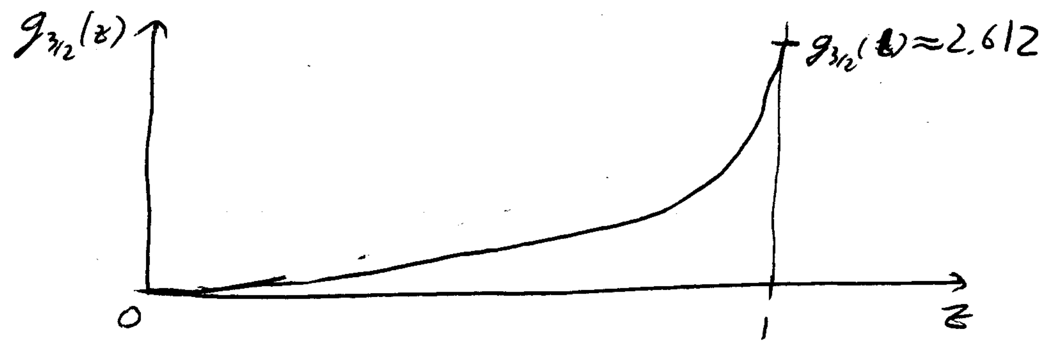
$$\langle n \rangle = \frac{1}{V}\frac{z}{1-z} + \frac{4}{\lambda_T^3\sqrt{\pi}}\int_0^\infty dx\, x^2 \frac{z}{e^{x^2}-z} - \underbrace{\frac{4}{\lambda_T^3\sqrt{\pi}}\int_0^{\lambda_T \frac{\sqrt{\pi}}{L}} dx\, x^2 \frac{z}{e^{x^2}-z}}_{\longrightarrow 0 \text{ for } L\to\infty \text{ for any } 0 \le z \le 1}$$

$$\frac{4}{\sqrt{\pi}}\int_0^\infty dx\, x^2 \frac{z}{e^{x^2}-z} = \frac{4z}{\sqrt{\pi}}\int_0^\infty dx\, x^2 e^{-x^2}\frac{1}{1-ze^{-x^2}}$$

$$= \frac{4}{\sqrt{\pi}} z \int_0^\infty dx\, x^2 \sum_{n=0}^{\infty} z^n e^{-(n+1)x^2} = \frac{4}{\sqrt{\pi}}\sum_{n=1}^{\infty} z^n \int_0^\infty dx\, x^2 e^{-nx^2}$$

$$= \frac{4}{\sqrt{\pi}}\sum_{n=1}^{\infty} z^n \left(-\frac{d}{dn}\int_0^\infty dx\, e^{-nx^2}\right) = \frac{4}{\sqrt{\pi}}\sum_{n=1}^{\infty} z^n \left(-\frac{d}{dn}\right)\frac{1}{2}\sqrt{\frac{\pi}{n}} = \sum_{n=1}^{\infty}\frac{z^n}{n^{3/2}} = g_{3/2}(z)$$

$$\langle n \rangle = \frac{1}{V}\frac{z}{1-z} + \frac{1}{\lambda_T^3} g_{3/2}(z) \quad \text{for large } V, \langle N \rangle$$



Two possible cases:

$\boxed{\text{low particle density} \quad n < \frac{1}{\lambda_T^3} g_{3/2}(1)}$

$\Rightarrow$ exists $0 < z_0 < 1$ such that $n = \frac{1}{\lambda_T^3} g_{3/2}(z_0)$

$$\Rightarrow \langle n \rangle = \frac{1}{V}\frac{z_0}{1-z_0} + \frac{1}{\lambda_T^3} g_{3/2}(z_0) \longrightarrow n \quad \text{for } V \to \infty$$

$\boxed{\text{high particle density} \quad n > \frac{1}{\lambda_T^3} g_{3/2}(1)}$

for any $z < 1$ $\langle n \rangle = \frac{1}{V}\frac{z}{1-z} + \underbrace{\frac{1}{\lambda_T^3} g_{3/2}(z)}_{< \frac{1}{\lambda_T^3} g_{3/2}(1) < n}$

$\Rightarrow \frac{1}{V}\frac{z}{1-z}$ has to give missing contribution

$$n - \frac{1}{\lambda_T^3} g_{3/2}(1) \equiv n_0$$

$\Rightarrow$ $z = z(V)$ given by $n_0 V = \frac{z(V)}{1-z(V)} \Rightarrow z(V) = 1 - \frac{1}{n_0 V}$

Then: $$\langle n \rangle = \underbrace{\frac{1}{V}\frac{z(V)}{1-z(V)}}_{= n_0} + \underbrace{\frac{1}{\lambda_T^3} g_{3/2}(\underbrace{z(V)}_{\xrightarrow[V\to\infty]{} 1})}_{\xrightarrow[V\to\infty]{} n - n_0} = n$$

Summary: (for $V \to \infty$ at fixed $n = \frac{N}{V}$)

$n < \frac{1}{\lambda_T^3} g_{3/2}(1) \Rightarrow z < 1$ given by $n = \frac{1}{\lambda_T^3} g_{3/2}(z)$

particle density in ground state $\frac{1}{V}\frac{z}{1-z} \to 0$

$n > \frac{1}{\lambda_T^3} g_{3/2}(1) \Rightarrow z = 1$, particle density in ground state

$\frac{1}{V}\frac{z}{1-z} = n_0 = n - \frac{g_{3/2}(1)}{\lambda_T^3}$ finite

"Bose - Einstein condensation"

At high densities or low temperatures a macroscopic number of particles condenses into the ground state.

Transition happens at

$$n = \frac{g_{3/2}(1)}{\lambda_T^3}$$

$\Rightarrow$ critical density at given temperature

$$n_c = \frac{g_{3/2}(1)}{\lambda_T^3} = 2.612 \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$$

critical temperature at given density

$$T_c = \frac{2\pi\hbar^2}{m k_B} \left(\frac{n}{g_{3/2}(1)}\right)^{2/3}$$

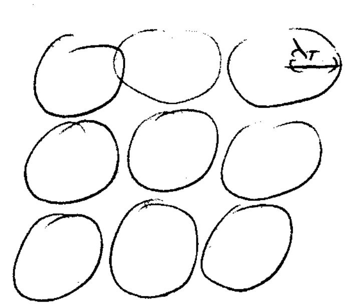
rule of thumb:

λ_T : quantum mechanical ~~extens~~ size of a particle at temperature T

λ_T^3 : quantum mechanical volume of a particle at temperature T

phase transition happens, when volume per particle is on the order of the quantum mechanical volume of the particle

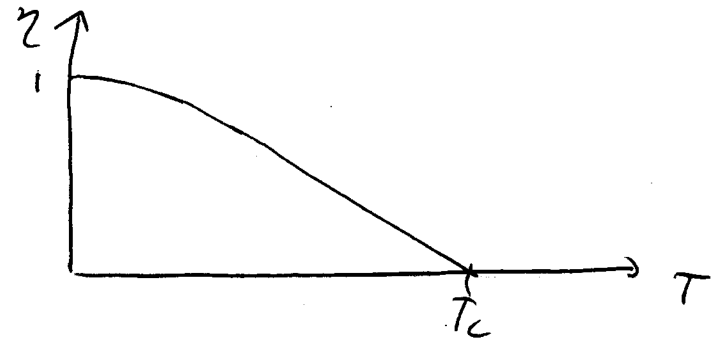
$$\frac{V}{N} \simeq \lambda_T^3$$



- *fraction of ground state particles:*

$$\eta \equiv \frac{n_0}{n} \overset{n>n_c}{=} \frac{n - \frac{g_{3/2}(1)}{\lambda_T^3}}{n} = 1 - \frac{g_{3/2}(1)}{n\lambda_T^3} = 1 - \frac{\lambda_{T_c}^3}{\lambda_T^3} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

$$\underset{n<n_c}{=} 0$$



pressure:

$$P = -\frac{\Omega}{V} = -\frac{k_B T \ln(1-z)}{V} - \frac{4 k_B T}{\sqrt{\pi}\,\lambda_T^3} \int_{\lambda_T \frac{\sqrt{\pi}}{L}}^{\infty} dx\, x^2 \ln(1 - e^{-x^2} z)$$

$$= -\frac{k_B T \ln(1-z)}{V} - \frac{4 k_B T}{\sqrt{\pi}\,\lambda_T^3} \int_0^{\infty} dx\, x^2 \ln(1-e^{-x^2} z) + \underbrace{\frac{4 k_B T}{\sqrt{\pi}\,\lambda_T^3} \int_0^{\lambda_T \frac{\sqrt{\pi}}{L}} dx\, x^2 \underbrace{\ln(1-e^{-x^2} z}_{\simeq \ln(1-z+zx}}_{\xrightarrow[L\to\infty]{} 0 \text{ for any } z<}$$

$$\frac{4}{\sqrt{\pi}} \int_0^{\infty} dx\, x^2 \ln(1-e^{-x^2} z) = -\frac{4}{\sqrt{\pi}} \sum_{n=1}^{\infty} \int_0^{\infty} dx\, x^2 \frac{1}{n} (e^{-x^2} z)^n$$

$$= -\frac{4}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{z^n}{n} \left(-\frac{d}{dn}\right) \int_0^{\infty} dx\, e^{-nx^2}$$

$$= -\frac{4}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{z^n}{n} \left(-\frac{d}{dn}\right) \frac{1}{2} \sqrt{\frac{\pi}{n}} = -\sum_{n=1}^{\infty} \frac{z^n}{n^{5/2}} = -g_{5/2}(z)$$

- $$P = \underbrace{-\frac{k_B T \ln(1-z)}{V}}_{\cancel{T} \geq T_c \Rightarrow z<1 \xrightarrow[V\to\infty]{} 0} + \frac{k_B T}{\lambda_T^3} g_{5/2}(z) \xrightarrow[V\to\infty]{} \boxed{P = \frac{k_B T}{\lambda_T^3} g_{5/2}(z)}$$

$$T > T_c \Rightarrow z = z(V) = 1 - \frac{1}{n_0 V} \quad \frac{\ln(1-z(V))}{V} = \frac{\ln \frac{1}{n_0 V}}{V} = -\frac{\ln n_0 + \ln V}{V} \xrightarrow[V\to\infty]{} 0$$

– P is constant in the condensed phase

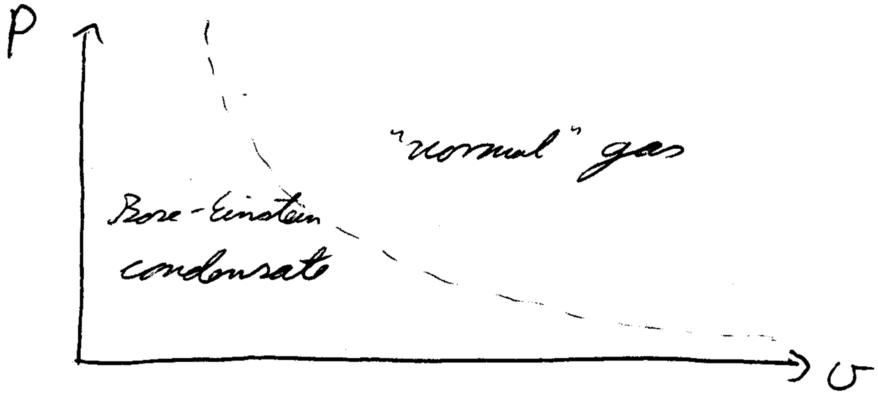
$$P = P_c = \frac{k_B T}{\lambda_T^3} g_{5/2}(1) \qquad \text{combine with} \qquad v_c = \frac{1}{n_c} = \frac{\lambda_T^3}{g_{3/2}(1)}$$

$$\Rightarrow \cancel{P_c} \qquad\qquad \Rightarrow g_{3/2}(1)\, v_c = \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$$

$$\frac{2\pi\hbar^2}{m k_B}\left(g_{3/2}(1)\, v_c\right)^{-2/3} = T$$

$$\cancel{\Rightarrow P_c = \frac{2\pi\hbar^2}{m}\left(g_{3/2}(1)\, v_c^{1/3}\right)^{1/3} g_{5/2}(1) \frac{v_c}{g_{3/2}(1)}}$$

$$P_c = \frac{2\pi\hbar^2}{m}\left(g_{3/2}(1)\, v_c\right)^{-2/3} \frac{g_{5/2}(1)}{v_c\, g_{3/2}(1)} = \frac{2\pi\hbar^2}{m} \frac{g_{5/2}(1)}{g_{3/2}(1)^{5/3}} \frac{1}{v_c^{5/3}}$$



entropy:

$$s = \frac{S}{V} = \left(\frac{\partial S}{\partial V}\right)_{T,\mu'} = \left(\frac{\partial P}{\partial T}\right)_{V,\mu'} \qquad P = \begin{cases} \frac{k_B T}{\lambda_T^3} g_{5/2}(z) & z<1 \\ \frac{k_B T}{\lambda_T^3} g_{5/2}(1) & z=1 \end{cases}$$

$$\lambda_T \sim T^{-1/2}$$

$$s = \begin{cases} k_B \frac{5}{2} \frac{1}{\lambda_T^3} g_{5/2}(z) + \frac{k_B T}{\lambda_T^3} \overbrace{g'_{5/2}(z)}^{\frac{g_{3/2}(z)}{z}} \overbrace{\left(\frac{\partial z}{\partial T}\right)_{V,\mu'}}^{\left(-\frac{z\mu'}{k_B T^2}\right)} & z<1 \\ k_B \frac{5}{2} \frac{1}{\lambda_T^3} g_{5/2}(1) & z=1 \end{cases}$$

$$= \begin{cases} k_B \frac{5}{2} \frac{1}{\lambda_T^3} g_{5/2}(z) - k_B \overbrace{\frac{g_{3/2}(z)}{\lambda_T^3}}^{\langle n \rangle} \ln z & z<1 \\ k_B \frac{5}{2} \frac{1}{\lambda_T^3} g_{5/2}(1) & z=1 \end{cases}$$