10. Response functions  
Compute the heat capacity at constant magnetic field \( C_{H,n} \), the susceptibilities \( \chi_T,n \) and \( \chi_S,n \), and the thermal expansivity \( \alpha_{H,n} \) for a magnetic system as functions of \( n \), \( M \), and \( T \) given that the mechanical equation of state is \( M = nDH/T \) and the heat capacity is \( C_{M,n} = nc \), where \( M \) is the magnetization, \( H \) is the magnetic field, \( n \) is the number of moles, \( D \) is a constant, \( c \) is the molar heat capacity, and \( T \) is the temperature.

11. Rubber band  
Experimentally one finds that for a rubber band

\[
\left( \frac{\partial J}{\partial L} \right)_T = \frac{aT}{L_0} \left( 1 + 2 \left( \frac{L_0}{L} \right)^3 \right) \quad \text{and} \quad \left( \frac{\partial J}{\partial T} \right)_L = \frac{aL}{L_0} \left( 1 - \left( \frac{L_0}{L} \right)^3 \right)
\]

where \( J \) is the tension, \( a = 10^{-2} N/K \), and \( L_0 = 0.5m \) is the length of the band at \( T_0 = 290K \) when no tension is applied. The mass of the rubber band is held fixed.

a) Compute \( (\partial L/\partial T)_J \) and discuss its physical meaning.

b) Show that \( dJ \) is an exact differential and find the equation of state \( J(L,T) \).

c) Show that the internal energy if interpreted as a function of temperature \( T \) and length \( L \) does not depend on \( L \). (Hint: The formula in problem 8 a) may be helpful.)

d) Assume that the heat capacity at constant length is \( C_L = 1J/K \). Find the change in temperature during stretching the band reversibly and adiabatically to a length of 1m and the work necessary to perform this stretching.
12. Osmotic pressure

How does a tree manage to get water up into its leaves? It makes use of the “osmotic pressure” that can act against gravity. The osmotic pressure arises if a sugar solution is in contact with pure water through a semi-permeable membrane, i.e., a membrane that can pass water molecules but not sugar molecules. We want to derive an expression for the osmotic pressure $\pi$. To this end we consider a setup of two chambers separated by a mechanically fixed semi-permeable membrane. One chamber contains pure water while the other chamber contains a sugar solution. Water molecules can pass through the semi-permeable membrane as they wish. Since the membrane is mechanically fixed, the pressure in the two chambers can be controlled independently and we will fix it at $P_0$ on the water side and at $P$ on the side of the sugar solution. This corresponds to an osmotic pressure of $\pi \equiv P - P_0$. We also keep the temperature $T$ of the system constant. We denote the number of moles of water molecules in the sugar solution chamber by $n_W$ and the number of moles of sugar molecules by $n_S$. Thus, the concentration of sugar in the sugar solution is $x_S = n_S/(n_S + n_W)$ and we will assume that this concentration is small. The Gibbs free energy for the chamber with the sugar solution is

$$G(P, T, n_S, n_W) = n_W \mu_W^{(0)}(P, T) + n_S \mu_S^{(0)}(P, T) - \lambda \frac{n_S n_W}{n_S + n_W} + n_W RT \ln(1 - x_S) + n_S RT \ln(x_S)$$

where $\mu_W^{(0)}(P, T)$ and $\mu_S^{(0)}(P, T)$ are the chemical potentials for water and sugar molecules by themselves, $\lambda$ is a constant describing the interaction between water and sugar molecules and the last two terms originate from the mixing.

a) Calculate the chemical potential $\mu_W(P, T, x_S)$ of the water molecules in the sugar solution.

b) Expand your result from a) for small $x_S$.

c) Write down the conditions for thermodynamic equilibrium between the two chambers.

d) Express the difference in chemical potentials $\mu_W^{(0)}(P, T) - \mu_W^{(0)}(P_0, T)$ in terms of the osmotic pressure $\pi$, the volume $V$ of the chamber with the sugar solution and the number $n_W$ of moles of water in the chamber with the sugar solution. Since water is very hard to compress you may assume that the volume $V$ does not change when you vary the pressure from $P_0$ to $P$ and when you add the sugar. (Hint: look at $(\partial G/\partial P)_T$.)

e) Combine your result into a relation between $\pi$, $V$, $n_S$, and $T$. This relation is called van’t Hoff’s law.