Department of Physics
The Ohio State University

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Fourth Problem Set for Physics 846 (Statistical Physics I)

Fall quarter 2003

Important dates: Oct 21 9:30am-10:18am make-up class,
Oct 30 10:30am-12:18pm midterm exam,
Nov 11 no class, Nov 27 no class, Dec 11 9:30am-11:18am final exam

Due date: Thursday, Oct 23

10. Response functions *8 points*

Compute the heat capacity at constant magnetic field $C_{H,n}$, the susceptibilities $\chi_{T,n}$ and $\chi_{S,n}$, and the thermal expansivity $\alpha_{H,n}$ for a magnetic system as functions of n, M, and T given that the mechanical equation of state is $M = nDH/T$ and the heat capacity is $C_{M,n} = nc$, where M is the magnetization, H is the magnetic field, n is the number of moles, D is a constant, c is the molar heat capacity, and T is the temperature.

11. Rubber band *12 points*

Experimentally one finds that for a rubber band

$$\left(\frac{\partial J}{\partial L}\right)_T = \frac{aT}{L_0}\left(1 + 2\left(\frac{L_0}{L}\right)^3\right) \quad \text{and} \quad \left(\frac{\partial J}{\partial T}\right)_L = \frac{aL}{L_0}\left(1 - \left(\frac{L_0}{L}\right)^3\right)$$

where J is the tension, $a = 10^{-2}N/K$, and $L_0 = 0.5m$ is the length of the band at $T_0 = 290K$ when no tension is applied. The mass of the rubber band is held fixed.

a) Compute $(\partial L/\partial T)_J$ and discuss its physical meaning.

b) Show that dJ is an exact differential and find the equation of state $J(L,T)$.

c) Show that the internal energy if interpreted as a function of temperature T and length L does not depend on L. (Hint: The formula in problem 8 a) may be helpful.)

d) Assume that the heat capacity at constant length is $C_L = 1J/K$. Find the change in temperature during stretching the band reversibly and adiabatically to a length of $1m$ and the work necessary to perform this stretching.

12. Osmotic pressure *10 points*

How does a tree manage to get water up into its leafs? It makes use of the "osmotic pressure" that can act against gravity. The osmotic pressure arises if a sugar solution is in contact with pure water through a semi-permeable membrane, i.e., a membrane that can pass water molecules but not sugar molecules. We want to derive an expression for the osmotic pressure π. To this end we consider a setup of two chambers separated by a mechanically fixed semi-permeable membrane. One chamber contains pure water while the other chamber contains a sugar solution. Water molecules can pass through the semi-permeable membrane as they wish. Since the membrane is mechanically fixed, the pressure in the two chambers can be controlled independently and we will fix it at P_0 on the water side and at P on the side of the sugar solution. This corresponds to an osmotic pressure of $\pi \equiv P - P_0$. We also keep the temperature T of the system constant. We denote the number of moles of water molecules in the sugar solution chamber by n_W and the number of moles of sugar molecules by n_S. Thus, the concentration of sugar in the sugar solution is $x_S = n_S/(n_S + n_W)$ and we will assume that this concentration is small. The Gibbs free energy for the chamber with the sugar solution is

$$G(P,T,n_S,n_W) = n_W \mu_W^{(0)}(P,T) + n_S \mu_S^{(0)}(P,T) - \lambda \frac{n_S n_W}{n_S + n_W} + n_W RT \ln(1 - x_S) + n_S RT \ln(x_S)$$

where $\mu_W^0(P,T)$ and $\mu_W^{(0)}(P,T)$ are the chemical potentials for water and sugar molecules by themselves, λ is a constant describing the interaction between water and sugar molecules and the last two terms originate from the mixing.

a) Calculate the chemical potential $\mu_W(P,T,x_S)$ of the water molecules in the sugar solution.

b) Expand your result from a) for small x_S.

c) Write down the conditions for thermodynamic equilibrium between the two chambers.

d) Express the difference in chemical potentials $\mu_W^{(0)}(P,T) - \mu_W^{(0)}(P_0,T)$ in terms of the osmotic pressure π, the volume V of the chamber with the sugar solution and the number n_W of moles of water in the chamber with the sugar solution. Since water is very hard to compress you may assume that the volume V does not change when you vary the pressure from P_0 to P and when you add the sugar. (Hint: look at $(\partial G/\partial P)_T$.)

e) Combine your result into a relation between π, V, n_S, and T. This relation is called van't Hoff's law.