Problem 22:

a) $f_Z(k) = \int_{-\infty}^{\infty} dz\, e^{ikz} P_Z(z)$

$$= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\ \delta(z-(x^2+y^2))\, P_X(x)\, P_Y(y)\, e^{ikz}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{ikx^2} e^{iky^2} P_X(x)\, P_Y(y) \quad (1)$$

$$= \left[\int_{-\infty}^{\infty} dx\, e^{ikx^2} P_X(x)\right]^2 = \left[\int_{-\infty}^{\infty} dx\, e^{ikx^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]^2$$

$$= \frac{1}{2\pi} \frac{\pi}{\frac{1}{2}-ik} = \frac{1}{1-2ik} \quad (1)$$

b) $\langle Z \rangle = \langle X^2+Y^2 \rangle = \langle X^2 \rangle + \langle Y^2 \rangle = 1+1 = 2 \quad (1)$

$$\langle Z^2 \rangle = \langle (X^2+Y^2)^2 \rangle = \langle X^4 \rangle + 2\langle X^2 Y^2 \rangle + \langle Y^4 \rangle$$

$$= 2\langle X^4 \rangle + 2\langle X^2 \rangle \langle Y^2 \rangle = 2\frac{4!}{2^{4/2}(4/2)!} + 2$$

$$= 2\frac{4\cdot 3\cdot 2}{2\cdot 2\cdot 2} + 2 = 8 \quad (1)$$

$$\langle Z^3 \rangle = \langle (X^2+Y^2)^3 \rangle = \langle X^6 \rangle + 3\langle X^2 Y^4 \rangle + 3\langle X^4 Y^2 \rangle + \langle Y^6 \rangle$$

$$= 2\langle X^6 \rangle + 6\langle X^2 \rangle \langle X^4 \rangle = 2\frac{6!}{2^{6/2}(6/2)!} + 6\cdot 1\cdot 3$$

$$= 2\frac{3\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 2\cdot 2\cdot 3\cdot 2} + 18 = 48 \quad (1)$$

also accept if derived from characteristic function

c) $C_1(Z) = \langle Z \rangle = 2 \quad (1)$

$C_2(Z) = \langle Z^2 \rangle - \langle Z \rangle^2 = 8 - 2^2 = 4 \quad (1)$

$C_3(Z) = \langle Z^3 \rangle - 3\langle Z \rangle \langle Z^2 \rangle + 2\langle Z \rangle^3$

$= 48 - 3\cdot 2\cdot 8 + 2\cdot 2^3 = 16 \quad (1)$

Problem 23:

a) $\langle Y_N \rangle = \langle \sum_{i=1}^{N} X_i \rangle = \sum_{i=1}^{N} \langle X_i \rangle = N \langle X_1 \rangle = N[p(-1)+(1-p)1] = N(1-2p)$

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b) $\sigma_{Y_N}^2 = \langle Y_N^2 \rangle - \langle Y_N \rangle^2 = \langle (\sum_{i=1}^{N} X_i)^2 \rangle - \langle \sum_{i=1}^{N} X_i \rangle \langle \sum_{i=1}^{N} X_i \rangle$

$$= \langle (\sum_{i=1}^{N} X_i)(\sum_{j=1}^{N} X_j) \rangle - \sum_{i=1}^{N} \langle X_i \rangle \sum_{j=1}^{N} \langle X_j \rangle$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \langle X_i X_j \rangle - \sum_{i=1}^{N} \sum_{j=1}^{N} \langle X_i \rangle \langle X_j \rangle$$

$$= \sum_{i=1}^{N} \langle X_i^2 \rangle + \sum_{i \neq j} \langle X_i \rangle \langle X_j \rangle - \sum_{i=1}^{N} \sum_{j=1}^{N} \langle X_i \rangle \langle X_j \rangle$$

$$= \sum_{i=1}^{N} \langle X_i^2 \rangle - \sum_{i=1}^{N} \langle X_i \rangle^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} \langle X_i \rangle \langle X_j \rangle - \sum_{i=1}^{N} \sum_{j=1}^{N} \langle X_i \rangle \langle X_j \rangle$$

$$= N[\langle X_1^2 \rangle - \langle X_1 \rangle^2]$$

$$= N[p(-1)^2 + (1-p) \cdot 1^2 - (1-2p)^2] = N[1 - 1 + 4p - 4p^2] = 4p(1-p)N$$

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c) $Y_N = 0 \Leftrightarrow$ as many up steps as down steps

$\Rightarrow \frac{N}{2}$ up steps and $\frac{N}{2}$ down steps

number of configurations $\binom{N}{\frac{N}{2}}$

probability of each configuration $p^{\frac{N}{2}}(1-p)^{\frac{N}{2}}$

$\Rightarrow \Pr\{Y_N = 0\} = \binom{N}{\frac{N}{2}} p^{\frac{N}{2}}(1-p)^{\frac{N}{2}}$ 4

d) $\Pr\{Y_N = 0\} = \frac{N!}{(\frac{N}{2})!(\frac{N}{2})!} p^{\frac{N}{2}}(1-p)^{\frac{N}{2}} \approx \frac{\sqrt{2\pi}\, N^{N+\frac{1}{2}} \cancel{e^{-N}}}{\sqrt{2\pi}^2 (\frac{N}{2})^{N+1} \cancel{e^{-N}}} p^{\frac{N}{2}}(1-p)^{\frac{N}{2}}$

$$= \frac{1}{\sqrt{2\pi}} \frac{N^N N^{\frac{1}{2}} 2^{N+1}}{N^N N} = \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{N}} (2p)^{\frac{N}{2}} [2(1-p)]^{N/2}$$

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