Problem 26:

2 a) Each of the N spins has 2 possible states
$\Rightarrow$ 2^N total states

2 b) $E = m\mu H$ with integer m. At this given E there are

$\frac{N-m}{2}$ spins "up" and $\frac{N+m}{2}$ spins "down".

The number of states with $\frac{N+M}{2}$ spins "up" is

$$N(E) = \binom{N}{\frac{N-m}{2}}$$

c) $$S = k_B \ln N(E) = k_B \ln \binom{N}{\frac{N-m}{2}} \quad 1$$

$$\approx k_B\left[N\ln N - \cancel{N}\ \frac{N+m}{2}\ln\frac{N+m}{2} + \cancel{\frac{N+m}{2}} - \frac{N-m}{2}\ln\frac{N-m}{2} + \cancel{\frac{N-m}{2}}\right]$$

$$= k_B\left[N\ln N - \left(\frac{N}{2} + \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2}+\frac{E}{2\mu H}\right) - \left(\frac{N}{2} - \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right)\right]$$

$$= k_B\left[\cancel{N\ln N - N\left(\frac{1}{2} + \frac{E}{2\mu HN}\right)\ln N} - N\left(\frac{1}{2} + \frac{E}{2\mu HN}\right)\ln\left(\frac{1}{2}+\frac{E}{2\mu HN}\right)\right.$$
$$\left. \cancel{- N\left(\frac{1}{2} - \frac{E}{2\mu HN}\right)\ln N} - N\left(\frac{1}{2} - \frac{E}{2\mu HN}\right)\ln\left(\frac{1}{2} - \frac{E}{2\mu HN}\right)\right]$$

$$= -Nk_B\left[\left(\frac{1}{2}+\frac{E}{2\mu HN}\right)\ln\left(\frac{1}{2}+\frac{E}{2\mu HN}\right) + \left(\frac{1}{2}-\frac{E}{2\mu HN}\right)\ln\left(\frac{1}{2}-\frac{E}{2\mu HN}\right)\right]$$

2

d) $$\frac{1}{T} \overset{①}{=} \left(\frac{\partial S}{\partial E}\right)_{N,H} = -Nk_B\left[\frac{1}{2\mu HN}\ln\left(\frac{1}{2}+\frac{E}{2\mu HN}\right) + \cancel{\frac{1}{2\mu HN}} - \frac{1}{2\mu HN}\ln\left(\frac{1}{2}-\frac{E}{2\mu HN}\right) - \cancel{\frac{1}{2\mu HN}}\right]$$

$$= -\frac{k_B}{2\mu H}\ln\frac{1+\frac{E}{\mu HN}}{1-\frac{E}{\mu HN}} \quad ②$$

e) $$\Rightarrow -\frac{2\mu H}{k_B T} = \ln\frac{1+\frac{E}{\mu HN}}{1-\frac{E}{\mu HN}}$$

$$e^{-\frac{2\mu H}{k_B T}} = \frac{1+\frac{E}{\mu HN}}{1-\frac{E}{\mu HN}} \quad ②$$

$$e^{-\frac{2\mu H}{k_B T}} - \frac{E}{\mu N} e^{-\frac{2\mu H}{k_B T}} = 1 + \frac{E}{\mu HN}$$

$$\frac{E}{\mu HN} = \frac{e^{-\frac{2\mu H}{k_B T}} - 1}{e^{-\frac{2\mu H}{k_B T}} + 1} = \frac{e^{-\frac{\mu H}{k_B T}} - e^{\frac{\mu H}{k_B T}}}{e^{-\frac{\mu H}{k_B T}} + e^{\frac{\mu H}{k_B T}}} = -\tanh\frac{\mu H}{k_B T}$$

$$E = -\mu HN \tanh\frac{\mu H}{k_B T}$$

f) $$M \overset{①}{=} -E/H = \mu N \tanh\frac{\mu H}{k_B T}$$

$$\chi_T = \left(\frac{\partial M}{\partial H}\right)_T = \mu N \frac{\mu}{k_B T}\left[1-\tanh^2\frac{\mu H}{k_B T}\right] \quad ②$$

g) $$\alpha = \left(\frac{\partial M}{\partial T}\right)_H = -\mu N \frac{\mu H}{k_B T^2}\left[1-\tanh^2\frac{\mu H}{k_B T}\right] \quad ②$$

h) insert e) into c): (1)

$$S = -N k_B \left\{ \left(\tfrac{1}{2} - \tfrac{1}{2}\tanh\frac{\mu H}{k_B T}\right) \ln\left(\tfrac{1}{2} - \tfrac{1}{2}\tanh\left(\frac{\mu H}{k_B T}\right)\right) + \left(\tfrac{1}{2} + \tfrac{1}{2}\tanh\left(\frac{\mu H}{k_B T}\right)\right) \ln\left[\tfrac{1}{2} + \tfrac{1}{2}\tanh\left(\frac{\mu H}{k_B T}\right)\right] \right\}$$

$$= -N k_B \left\{ \frac{e^{\frac{\mu H}{k_B T}} + e^{-\frac{\mu H}{k_B T}} - e^{\frac{\mu H}{k_B T}} + e^{-\frac{\mu H}{k_B T}}}{2 e^{\frac{\mu H}{k_B T}} + 2 e^{-\frac{\mu H}{k_B T}}} \ln \frac{e^{-\frac{\mu H}{k_B T}}}{e^{\frac{\mu H}{k_B T}} + e^{-\frac{\mu H}{k_B T}}} + \frac{e^{\frac{\mu H}{k_B T}}}{e^{\frac{\mu H}{k_B T}} + e^{-\frac{\mu H}{k_B T}}} \ln \frac{e^{\frac{\mu H}{k_B T}}}{e^{\frac{\mu H}{k_B T}} + e^{-\frac{\mu H}{k_B T}}} \right\}$$

$$= -N k_B \left\{ \beta\mu H \frac{e^{\beta\mu H} - e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}} - \ln\left[e^{\beta\mu H} + e^{-\beta\mu H}\right] \right\}$$

$$= -N k_B \left\{ \beta\mu H \tanh\beta\mu H - \ln\left[e^{\beta\mu H} + e^{-\beta\mu H}\right] \right\} \quad (2)$$

i) $$C_H \overset{(1)}{=} T\left(\frac{\partial S}{\partial T}\right)_{H,N} = T\left(\frac{\partial S}{\partial \beta}\right)_{H,N}\left(\frac{\partial \beta}{\partial T}\right)_{H,N} = -T\frac{1}{k_B T^2}\left(\frac{\partial S}{\partial \beta}\right)_{H,N}$$

$$\underset{(1)}{=} -\beta\left(\frac{\partial S}{\partial \beta}\right)_{H,N} = +N k_B \beta\mu H \tanh\beta\mu H + N k_B (\beta\mu H)^2\left[1 - \tanh^2\beta\mu H\right] - N k_B \beta\mu H \frac{e^{\beta\mu H} - e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}}$$

$$= N k_B (\beta\mu H)^2 \left[1 - \tanh^2(\beta\mu H)\right] \quad (2)$$

or $$C_H \overset{(1)}{=} \left(\frac{\partial E}{\partial T}\right)_{H,V} \overset{(1)}{=} -\mu H N \frac{-\mu H}{k_B T^2}\left[1 - \tanh^2(\beta\mu H)\right]$$

$$= N k_B (\beta\mu H)^2 \left[1 - \tanh^2(\beta\mu H)\right] \quad (2)$$

Problem 27:

a) Divide system into two subsystems 1 and 2

$p_i^{(1)}$	$p_j^{(2)}$
1	2
$S_n^{(1)}$ M_1	$S_n^{(2)}$ M_2

States in subsystem 1: i with probability $p_i^{(1)}$
states in subsystem 2: j with probability $p_j^{(2)}$

state in total system: (i,j) with probability $p_i^{(1)} p_j^{(2)}$

$$S_n \underset{1}{=} -\ln \sum_{(i,j)} \left[p_i^{(1)} p_j^{(2)}\right]^n = -\ln \sum_{i=1}^{M_1} \sum_{j=1}^{M_2} \left[p_i^{(1)}\right]^n \left[p_j^{(2)}\right]^n$$

$$= -\ln \left[\sum_{i=1}^{M_1} \left[p_i^{(1)}\right]^n\right]\left[\sum_{j=1}^{M_2} \left[p_j^{(2)}\right]^n\right]$$

$$= -\ln \sum_{i=1}^{M_1} \left[p_i^{(1)}\right]^n - \ln \sum_{j=1}^{M_2} \left[p_j^{(2)}\right]^n \underset{2}{=} S_n^{(1)} + S_n^{(2)}$$

b) Maximizing S_n is the same as minimizing $\sum_{i=1}^{N} p_i^n$.

If energy E is given only those states i with $E_i = E$ have a probability that is not zero, i.e. $p_i = 0$ for $E_i \neq E$. There are $N(E)$ states with $E_i = E$. Order states such that these states are $i = 1, \ldots, N(E)$. Then we have to maximize $\sum_{i=1}^{N(E)} p_i^n$ under the constraint $\sum_{i=1}^{N(E)} p_i = 1$ 2

$$F(\{p_i\}) = \sum_{i=1}^{N(E)} p_i^n \qquad G(\{p_i\}) = \sum_{i=1}^{N(E)} p_i - 1$$

$$0 = \frac{d}{dp_i}(F + \lambda G) = n p_i^{n-1} + \lambda \Rightarrow p_i = \left(-\frac{\lambda}{n}\right)^{\frac{1}{n-1}} \neq 0$$

$\Rightarrow p_i$ 2 constant for $i = 1, \ldots, N(E)$

$$\Rightarrow p_i = \begin{cases} \frac{1}{N(E)} & E_i = E \\ 0 & E_i \neq E \end{cases} \quad 1$$