The moments are retrieved from the characteristic function by differentiating:

$$\langle X^n \rangle = \lim_{k\to 0} (-i)^n \frac{d^n f_X(k)}{dk^n}$$

Instead of moments often look at cumulants $C_n(X)$

$$f_X(k) = \exp\left(\sum_{n\geq 1} \frac{(ik)^n}{n!} C_n(X) \right)$$

$$C_1(X) = \langle X \rangle$$
$$C_2(X) = \langle X^2 \rangle - \langle X \rangle^2 = \sigma_X^2$$
$$C_3(X) = \langle X^3 \rangle - 3\langle X \rangle \langle X^2 \rangle + 2\langle X \rangle^3$$
$$C_4(X) = \langle X^4 \rangle - 3\langle X^2 \rangle^2 - 4\langle X \rangle\langle X^3 \rangle + 12\langle X \rangle^2\langle X^2 \rangle - 6\langle X \rangle^4$$

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Example: $P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

$$\langle X \rangle = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = 0 \qquad X_p = 0 \qquad X_m = 0$$

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{-1}{\sqrt{2\pi\sigma^2}} \left.\frac{d}{da}\right|_{a=\frac{1}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-ax^2} dx = -\frac{1}{\sqrt{2\pi\sigma^2}} \left.\frac{d}{da}\right|_{a=\frac{1}{2\sigma^2}} \sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\sigma^2}} \left(\frac{1}{2\sigma^2}\right)^{-\frac{3}{2}} = \sigma^2$$

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sigma$$

$$f_X(k) = \int_{-\infty}^{\infty} e^{ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-k^2\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-ik\sigma^2)^2} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2}\, e^{-k^2\frac{\sigma^2}{2}} = e^{-k^2\frac{\sigma^2}{2}}$$

Expand in k:

$$f_X(k) = \sum_{m=0}^{\infty} \frac{\left(-\frac{k^2\sigma^2}{2}\right)^m}{m!} \Rightarrow \sum_{n=0}^{\infty} \frac{(ik)^n \langle X^n \rangle}{n!}$$

$$\Rightarrow \langle X^n \rangle = \begin{cases} 0 & n \text{ odd} \\ \frac{\sigma^n}{2^{n/2}} \frac{n!}{(n/2)!} & n \text{ even} \end{cases}$$

Expand exponent in k:

$$f_X(k) = \exp\left[-k^2\frac{\sigma^2}{2}\right] = \exp\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n(X)\right)$$

$$\Rightarrow \quad C_n(X) = \begin{cases} \sigma^2 & n = 2 \\ 0 & n \neq 2 \end{cases}$$

Jointly distributed stochastic variables

$X_1, X_2, \ldots, X_n$ are jointly distributed if they are defined on the same sample space S.

joint distribution function:

$$F_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \text{Prob}\,\underbrace{\{X_1 < x_1, \ldots, X_n < x_n\}}$$

$$\equiv \{X_1 < x_1\} \cap \{X_2 < x_2\} \cap \ldots \cap \{X_n < x_n\}$$

joint probability density:

$$P_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \frac{\partial^n F(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

properties (for two variables X and Y)

$$F_{X,Y}(-\infty,y) = F_{X,Y}(x,-\infty) = F_{X,Y}(-\infty,-\infty) = 0$$
$$F_{X,Y}(+\infty,+\infty) = 1$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, P_{X,Y}(x,y) = 1$$

$$F_X(x) = F_{X,Y}(x,\infty) = \int_{-\infty}^{x} dx' \int_{-\infty}^{\infty} dy' \, P_{X,Y}(x',y')$$

$$P_X(x) = \int_{-\infty}^{\infty} P_{X,Y}(x,y)\,dy \qquad P_Y(y) = \int_{-\infty}^{\infty} P_{X,Y}(x,y)\,dx$$

<u>joint moments:</u>

$$\langle X^m Y^n \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; x^m y^n P_{X,Y}(x,y)$$

<u>covariance:</u>

$$\mathrm{Cov}(X,Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle = \langle XY \rangle - \langle X \rangle \langle Y \rangle$$

<u>correlation function</u>

$$\mathrm{Cor}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_x \sigma_y} = \frac{\langle XY \rangle - \langle X \rangle \langle Y \rangle}{\sigma_x \sigma_y}$$

properties:

- $\mathrm{Cor}(X,Y) = \mathrm{Cor}(Y,X)$
- $-1 \le \mathrm{Cor}(X,Y) \le 1$
- $\mathrm{Cor}(X,X) = 1$; $\mathrm{Cor}(X,-X) = -1$
- $\mathrm{Cor}(aX+b, cY+d) = \mathrm{Cor}(X,Y)$ if $a,c \neq 0$

X and Y are <u>independent</u> if $P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$

properties of independent variables:

- $\langle XY \rangle = \langle X \rangle \langle Y \rangle$
- $\mathrm{Cor}(X,Y) = 0$

II.1.3 Two important theorems

Central limit theorem

„Nearly all important distributions are Gaussian."

N statistically independent identically distributed (i.i.d.) random variables X_i with finite moments.

$$Y_N = \frac{1}{N}(X_1 + \dots + X_N) \quad \text{average}$$

What is the distribution of Y_N?

Assume without loss of generality $\langle X_i \rangle = 0$.

Generating function

$$f_{Y_N}(k) = \int_{-\infty}^{\infty} dy\, e^{iky} \underbrace{P_{Y_N}(y)}$$

$$= \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N\, \delta(y - \tfrac{1}{N}(x_1 + \dots + x_N))\, P_{X_1}(x_1) \cdot \dots \cdot P_{X_N}(x_N)$$

$$= \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N\, e^{ik/N(x_1 + \dots + x_N)}\, P_{X_1}(x_1) \dots P_{X_N}(x_N)$$

$$= \left[\int_{-\infty}^{\infty} dx\, e^{ik/N\, x} P_X(x)\right]^N = \left[1 - \frac{k^2}{2N^2}\sigma_X^2 + \frac{k^4}{4!N^4}\langle x^4 \rangle \mp \dots\right]^N$$

$$\simeq e^{-\frac{k^2}{2N\sigma_X^2}} \quad \text{for large } N.$$

$$(1 + x/N)^N \xrightarrow[N\to\infty]{} e^x$$

$$\Rightarrow P_{Y_N}(y) \xrightarrow[N\to\infty]{} \int_{-\infty}^{\infty} dk\, e^{iky} e^{-\frac{k^2}{2N\sigma_X^2}} = \sqrt{\frac{N}{2\pi\sigma_X^2}} \exp\left(-\frac{Ny^2}{2\sigma_X^2}\right)$$

Gaussian distribution with $\langle Y_N \rangle = 0$ and $\sigma_{Y_N} = \frac{1}{\sqrt{N}}\sigma_X$.

The law of large numbers

„If we average over enough independent contributors every observable takes on its average value"

X stochastic variable with $\sigma_X < \infty$
$X_1, -, X_N$ i.i.d. variables distributed like X
$Y_N = \frac{1}{N}(X_1 + \dots + X_N)$

Choose $\varepsilon > 0$. Want to show $\lim_{N\to\infty} \Pr\{|Y_N - \langle X\rangle| \geq \varepsilon\} = 0$

$$\sigma_{Y_N}^2 = \int_{-\infty}^{\infty} dy\,(y - \langle Y_N\rangle)^2\, P_{Y_N}(y)$$
$$\geq \int_{-\infty}^{\langle Y_N\rangle-\varepsilon} dy\,(y - \langle Y_N\rangle)^2 P_{Y_N}(y) + \int_{\langle Y_N\rangle+\varepsilon}^{\infty} dy\,(y - \langle Y_N\rangle)^2 P_{Y_N}(y)$$
$$\geq \varepsilon^2\left[\int_{-\infty}^{\langle Y_N\rangle-\varepsilon} dy\, P_{Y_N}(y) + \int_{\langle Y_N\rangle+\varepsilon}^{\infty} dy\, P_{Y_N}(y)\right] = \varepsilon^2 \Pr\{|Y_N - \langle Y_N\rangle| \geq \varepsilon\}$$
$$= \varepsilon^2 \Pr\{|Y_N - \langle X\rangle| \geq \varepsilon\}$$

$$\Rightarrow \Pr\{|Y_N - \langle X\rangle| \geq \varepsilon\} \leq \frac{\sigma_{Y_N}^2}{\varepsilon^2} = \frac{\sigma_X^2}{N\varepsilon^2} \xrightarrow[N\to\infty]{} 0$$

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