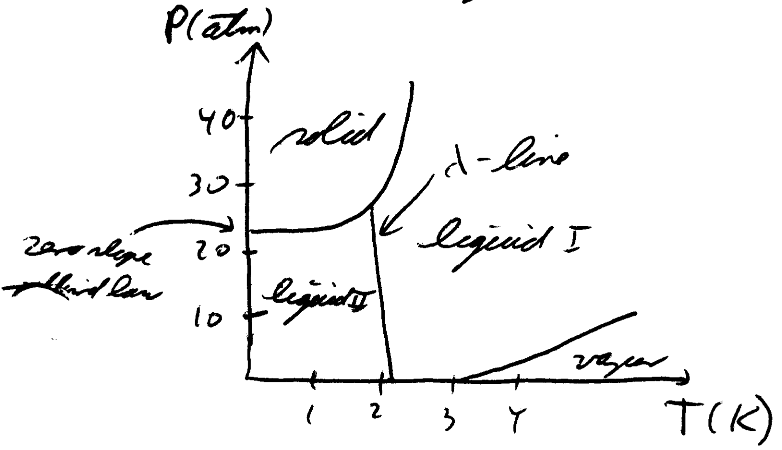
II. 4.3 Helium liquids

11/13

"Helium is special"
two types of helium: He^3 and He^4

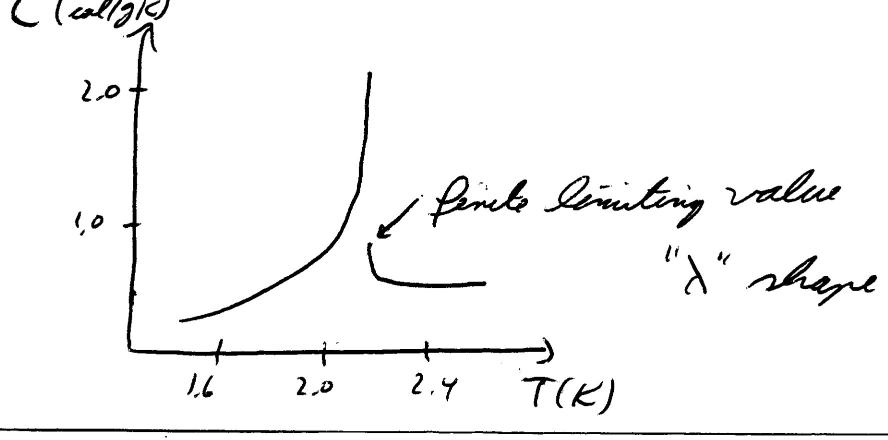
He^4

Phase diagram



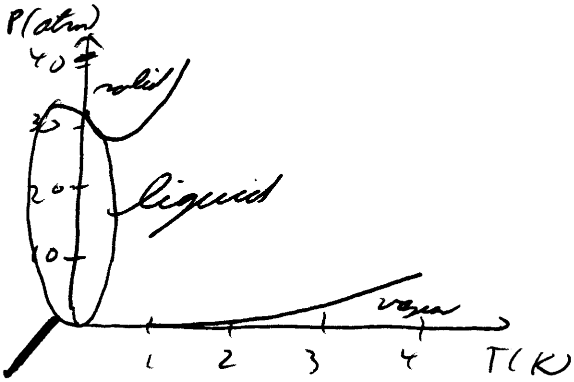
Unusual properties:

- liquid phase exists down to $T=0$
- two liquid phases
- λ-line is a line of continuous phase transitions between the two liquids
- liquid II is "superfluid", i.e., it flows without friction

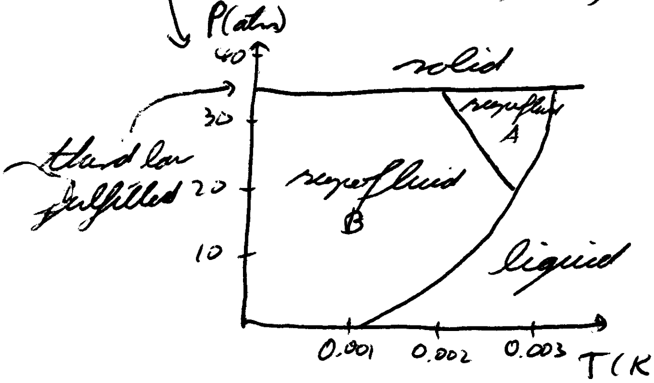


He^3

Phase diagram:



no superfluid phase?
third law not fulfilled?

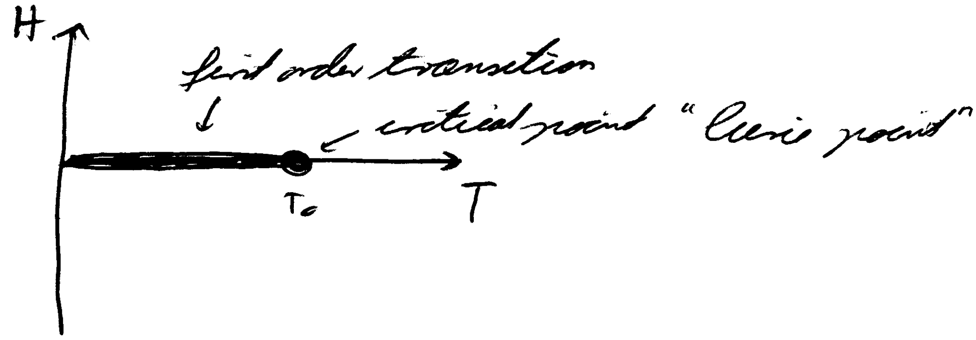


facts:

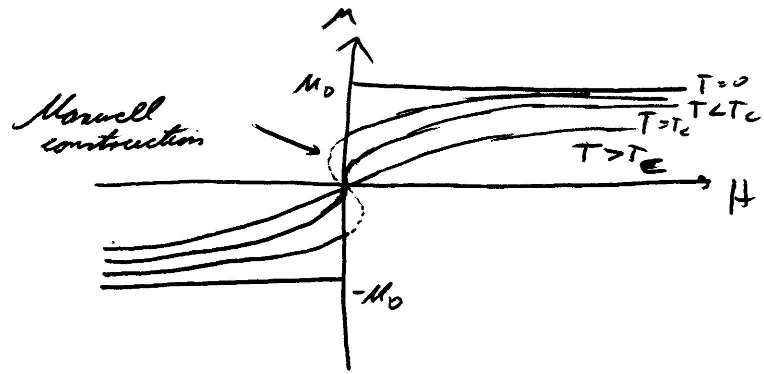
- superfluid A phase is anisotropic
- superfluid B phase is isotropic
- liquid - superfluid transitions continuous
- all other transitions first order

II.4.4 Magnetic systems

Phase diagram



M-H-diagram



very similar to liquid-gas transition

will derive phase diagram from microscopic model later

II.5 Summary

concepts: phase transition
first order / continuous / n-th order / infinite order
triple point
critical point
metastable states
law of corresponding states

facts: Gibbs phase rule
example phase diagrams

tools: Clausius-Clapeyron equation
Maxwell construction

III. Statistical description of physical phenomena

QM: everything is inherently statistical

classical physics: in principle the state of the system is known at every time if initial conditions are known.
In practice initial conditions never exactly known.

$\Rightarrow$ need probabilistic description

III.1 Introduction to probability and stochastic variables

III.1.1 Probability

Probability quantifies our expectation of a certain outcome.

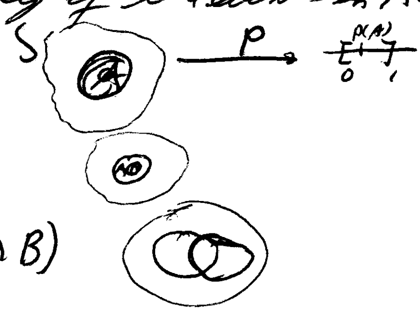
Start with (very large) sample space S of "possible events".
For each set $A \subset S$ $P(A)$ is the probability of the "events in A".

conditions: "$\text{Prob}(A) = \text{Prob}\{A\}$"

(i) $P(\emptyset) = 0$ $\quad P(S) = 1$

(ii) $A \subset B \Rightarrow P(A) \leq P(B)$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$P(A \cap B)$: probability that "A" and "B" occur together

$P(A \cup B)$: probability that "A" or "B" occurs

III. Statistical description of physical phenomena

QM: everything is inherently statistical
classical physics: in principle the state of a system is known at every time if initial condition are known.
In practice initial conditions never known exactly
$\Rightarrow$ need probabilistic description

III.1 Introduction to probability and stochastic variables

III.1.1 Probability (reading assignment)

- setup
sample space S, events $A \subset S$, probability $P(A) \in [0,1]$

- independent vs ~~mutually~~ exclusive
A and B are mutually exclusive if $A \cap B = \phi$
"A and B never occur at the same time"
A and B are independent if $P(A \cap B) = P(A)\,P(B)$
"the outcome of A is independent of the outcome of B"
mutually exclusive events are not independent (unless $P(A)=0$ or $P(B)=0$)

- $P(A|B)$ conditional probability of A given B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ (different from book)

if A and B are independent $P(A|B) = P(A)$

- ABRACADABRA $\qquad \dfrac{11!}{\underset{A}{5!}\,\underset{R}{2!}\,\underset{B}{2!}\,\underset{D}{1!}\,\underset{C}{1!}}$

Definitions:

$A_1, A_2, \dots, A_m$ are mutually exclusive if $A_i \cap A_j = \phi$ for all $i \neq j$

$A_1, A_2, \dots, A_m$ are exhaustive if $A_1 \cup A_2 \cup \dots \cup A_m = S$

A, B are independent if $P(A \cap B) = P(A)P(B)$

The conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$P(A|B)$ is the probability to draw A under the condition that B has been observed

III.1.2 Stochastic variables

A stochastic variable X is a function that maps elements of S into the real numbers

→ "observable"

continuous stochastic variable:

- takes a continuous set of variables
- characterized by probability density $P_X(x)$

defined by ↓ this is a subset of S

$$\mathrm{Prob}\{a \leq X \leq b\} = \int_a^b P_X(x)\,dx$$

$P_X(x)\,dx$ is the probability to find the stochastic variable in the interval $[x, x+dx]$

properties:

- $P_X(x) \geq 0$
- $\int_{-\infty}^{\infty} P_X(x) = 1$

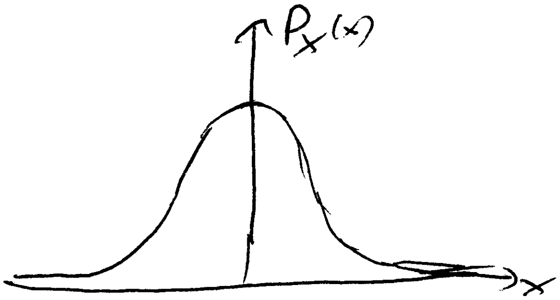
- equivalently characterized by <u>distribution function</u>

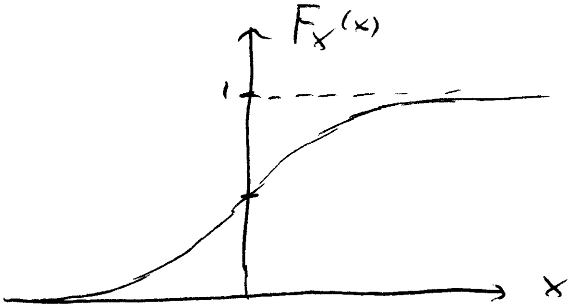
$$F_X(x) = \text{Prob}(X \leq x) = \int_{-\infty}^{x} dy\, P_X(y) \qquad P_X(x) = \frac{dF_X(x)}{dx}$$

properties:
- $F_X(x)$ increases monotonously
- $F_X(-\infty) = 0$
- $F_X(+\infty) = 1$

- example: Gaussian probability density

$$P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$





<u>discrete stochastic variable</u>
- takes countable number $x_1, x_2, \ldots$ of values
- characterized bei probabilities $p_1, p_2, \ldots$ $p_i = P(X^{-1}(\{x_i\}))$ to obtain these values

properties:
- $p_i \geq 0$
- $\sum_i p_i = 1$

- probability density function

$$P_X(x) = \sum_i p_i\, \delta(x - x_i)$$

- distribution function

$$F_X(x) = \sum_i p_i\, \Theta(x - x_i) \qquad \Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

moments and characteristic function

Definitions:

expectation value: $\langle f(X)\rangle \equiv \int_{-\infty}^{\infty} f(x)\, P_X(x)\, dx$

n-th moment: $\langle X^n\rangle = \int_{-\infty}^{\infty} x^n P_X(x)\, dx$

mean: $\langle X\rangle = \int_{-\infty}^{\infty} x\, P_X(x)\, dx$

most probable value: x_p: $P_X(x_p) \geq P_X(x)$ for all $x \in R$

median: x_m: $F_X(x_m) = \frac{1}{2}$

variance: $\langle (X - \langle X\rangle)^2\rangle = \langle X^2\rangle - \langle X\rangle^2$

standard deviation: $\sigma_X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$

characteristic function:

$$f_X(k) = \langle e^{ikX}\rangle = \int_{-\infty}^{\infty} e^{ikx} P_X(x)\, dx = \sum_{n=0}^{\infty} \frac{(ik)^n \langle x^n\rangle}{n!}$$

properties:

- $f_X(0) = 1$
- $|f_X(k)| \leq 1$
- $f_X(-k) = f_X^*(k)$ ← complex conjugation

$$P_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx} f_X(k)$$

"the characteristic function contains the same information as the probability density"

The moments are retrieved from the characteristic function by differentiating:

$$\langle X^n \rangle = \lim_{k\to 0} (-i)^n \frac{d^n f_X(k)}{dk^n}$$

Instead of moments often look at cumulants $C_n(X)$

$$f_X(k) = \exp\left(\sum_{n=1}^{\infty} \frac{(ik)^n}{n!} C_n(X) \right)$$

$$C_1(X) = \langle X \rangle$$
$$C_2(X) = \langle X^2 \rangle - \langle X \rangle^2 = \sigma_X^2$$
$$C_3(X) = \langle X^3 \rangle - 3\langle X \rangle \langle X^2 \rangle + 2\langle X \rangle^3$$
$$C_4(X) = \langle X^4 \rangle - 3\langle X^2 \rangle^2 - 4\langle X \rangle \langle X^3 \rangle + 12 \langle X \rangle^2 \langle X^2 \rangle - 6\langle X \rangle^4$$

JL 11/13

<u>Example:</u> $P_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$

$$\langle X \rangle = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = 0 \qquad X_p = 0 \qquad X_m = 0$$

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx = \frac{-1}{\sqrt{2\pi\sigma^2}} \frac{d}{da}\bigg|_{a=\frac{1}{2\sigma^2}} \int_{-\infty}^{\infty} e^{-ax^2} dx = -\frac{1}{\sqrt{2\pi\sigma^2}} \frac{d}{da}\bigg|_{a=\frac{1}{2\sigma^2}} \sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\sigma^2}} \left(\frac{1}{2\sigma^2}\right)^{-\frac{3}{2}} = \sigma^2$$

$$\sigma_X = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sigma$$

$$f_X(k) = \int_{-\infty}^{\infty} e^{ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-k^2\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - ik\sigma^2)^2} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2}\, e^{-k^2\frac{\sigma^2}{2}} = e^{-k^2\frac{\sigma^2}{2}}$$