Example:

\[ U(x) = -\frac{W d^2 (x^2 + d^2)}{x^4 + 8d^4} \]

1) note: \( x \) has dimension "length" 
   "d has dimension "length"

   = introduce dimensionless variable \( \eta = \frac{x}{d} \)
   "W has dimension "energy"

   = introduce dimensionless potential \( Z(\eta) = \frac{U(x)}{W} \)

\[ Z(\eta) = -\frac{\eta^2 + 1}{\eta^4 + 8} \]

2) equilibrium value, \( 0 = Z'(\eta) = \frac{-2\eta(\eta^4 + 8) + (\eta^2 + 1)^4 \eta^2}{(\eta^4 + 8)^2} = +2 \frac{\eta^4 + 2\eta^2 - 8}{(\eta^4 + 8)^2} \)

\[ \eta_0 = 2.0 \]
\[ \Rightarrow x_{eq} = 2d \]
\[ x_{eq} = \sqrt{2} d \]
\[ x_{eq} = -\sqrt{2} d \]

\[ Z(\eta) \to 0 \text{ as } \eta \to \pm \infty \]
\[ \frac{\partial Z(\eta)}{\partial \eta} \to 0 \text{ as } \eta \to \pm \infty \]

\[ x_{eq} = 0 \text{ unstable} \]
\[ x_{eq} = \sqrt{2} d, x_{eq} = -\sqrt{2} d \text{ stable} \]
Type of motion

\( E > 0 \) unbounded

\(-1/8 < E \leq 1/8 \) bounded oscillation around 0

\(-1/4 < \frac{E}{w} < 1/4 \) bounded oscillation around 0 at \( V_1 \) and \( -V_1 \)

\( \frac{E}{w} = \frac{1}{4} \) rest at either \( V_1d \) or \( -V_1d \)

\( \frac{E}{w} < \frac{1}{4} \) not possible

What happens at \( \frac{E}{w} = \frac{1}{8} \)?

3 possible motions:

a) rest at \( x = 0 \)

b) going off to \( x < 0 \) until \(-1/8 = -\frac{y^2 + 1}{y^2 + 8} \) \( \Rightarrow x = -2V_1d \)

\( y = -2V_1d \)

goes right

infinite slow approach to \( x = 0 \) (take force 0)

c) mirror image of b)
1.2.1 One-dimensional oscillation around still equilibrium point

What happens close to the still equilibrium point?

\[ U(x) = U(x^*) + \frac{dU}{dx}(x^*) (x-x^*) + \frac{1}{2} \frac{d^2U}{dx^2}(x^*) (x-x^*)^2 + O((x-x^*)^3) \]

\[ \Rightarrow 0 \quad \text{because } x^* \text{ is still equilibrium point} \]

Define \( k = \frac{d^2U}{dx^2}(x^*) > 0 \)

\[ y = x - x^* \]

\[ V(y) = U(x^* + y) - U(x^*) \]

\[ V(y) = \frac{1}{2}ky^2 \]

\[ \Rightarrow m\ddot{y} = -\frac{d}{dy}V(y) = -ky \]

\[ \ddot{y} + \omega_0^2 y = 0 \quad \text{with} \quad \omega_0^2 = \frac{k}{m} \]

\[ y(t) = A e^{\lambda t} \]

\[ A\lambda^2 e^{\lambda t} + \omega_0^2 A e^{\lambda t} = 0 \Rightarrow \lambda^2 + \omega_0^2 = 0 \Rightarrow \lambda = \pm i\omega_0 \]

\[ \Rightarrow \text{general solution:} \]

\[ y(t) = B e^{-\lambda t} \left\{ A \cos \omega_0 t + B \sin \omega_0 t \right\} \]

Another way of writing this

\[ y(t) = A \cos(\omega_0 t - \phi) \]

or \[ y(t) = A \sin(\omega_0 t - \phi) \]

\( \phi \) or \( \phi \) is the phase of the motion.

\( \omega_0 \) is the angular frequency of the motion.

\( A \) is the amplitude of the motion.