Free-free motion of a symmetric top

\[ I_1 = I_2 \neq I_3 \]

\[ \dot{\Omega}_3 = 0 \Rightarrow \Omega_3 = \text{const.} \]

\[ \dot{\Omega}_1 = \left( \frac{I_1 - I_3}{I_1} \right) \Omega_2 , \quad \dot{\Omega}_2 = \left( \frac{I_1 - I_3}{I_1} \right) \Omega_1 \]

\[ \Omega_1(t) = A \cos \Omega t \]

\[ \Omega_2(t) = A \sin \Omega t \]

\[ \Rightarrow \bar{\omega} \text{ rotates (in the body coordinate system) around the symmetry axis with frequency } \Omega \]

\[ \text{while } \dot{\bar{\omega}} \text{ is constant} \]

What does this mean in the fixed coordinate system?

- No forces \( \Rightarrow \bar{L} = \text{const} \) and \( \bar{L} = \frac{1}{2} \bar{\omega} \times \bar{L} \text{ constant} \)

- The angle between \( \bar{\omega} \) and \( \bar{L} \) is constant

How are \( \bar{L} \), \( \bar{\omega} \) and \( \bar{L}_3 \) related to each other?

\[ \bar{L} \times \bar{L}_3 = \omega_2 \bar{L} - \omega_3 \bar{L}_3 \]

\[ \Rightarrow \bar{L} \cdot (\bar{L} \times \bar{L}_3) = \omega_2 \bar{L} \bar{L}_3 - \omega_3 \bar{L} \bar{L}_3 = \omega_2 \omega_3 - \omega_3 \omega_2 \bar{L}_3 = 0 \]

\[ \Rightarrow \bar{L}, \bar{\omega}, \bar{L}_3 \text{ are always in the same plane} \]

\[ \bar{L} \text{(fixed in space)} \]

- the "body cone" rolls on the "spatial cone"
E.g. Earth
- not quite a sphere but flattened at the poles
\[ I_3 > I_2 = I_1 \]
\[ \Omega = \frac{I_1 - I_3}{I_1} \]
\[ \text{number gives } \Omega = \frac{1}{3 \text{ days}} \]
- actually \( \Omega = \frac{1}{260000} \) (Earth not rigid)

- "pole star is slowly changing"
Symmetric top invariants (recall readers assigned)

1. Lots of algebra to get to.

\[ T = \frac{1}{2} I_1 (\omega_x^2 + \omega_y^2) + \frac{1}{2} I_3 \omega_z^2 \]
\[ \omega_z = \dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{\phi}, \dot{\theta}, \dot{\psi} \]

\[ \rightarrow T = \frac{1}{2} I_1 (\dot{\phi} \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 \]

\[ \rightarrow L = \frac{1}{2} I_1 (\dot{\phi} \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos \theta + \dot{\psi})^2 - Mgh \cos \theta \]

Symmetric tensor quantities \( \mathbf{\bar{I}} \)

\[ \frac{\partial L}{\partial \dot{\phi}} = 0 \rightarrow \text{conserved} \]
\[ \frac{\partial L}{\partial \dot{\theta}} = 0 \quad (\phi \text{ cyclic}) \rightarrow \text{P}_\phi \text{ conserved} \]
\[ \frac{\partial L}{\partial \dot{\psi}} = 0 \quad (\theta \text{ cyclic}) \rightarrow \text{P}_\psi \text{ conserved} \]
\[ \rightarrow \text{MC+1} \]

\[ \text{P}_\phi = \frac{\partial L}{\partial \dot{\phi}} = (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\phi} + I_3 \dot{\psi} \cos \theta \]

\[ \text{P}_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 (\dot{\psi} + \dot{\phi} \cos \theta) = I_3 \omega_z \]

\[ \Rightarrow \dot{\phi} = \frac{\text{P}_\phi - \text{P}_\psi \cos \theta}{I_1 \sin^2 \theta}, \quad \dot{\psi} = \frac{\text{P}_\psi - (\text{P}_\phi - \text{P}_\psi \cos \theta) \cos \theta}{I_3} \]

\[ \text{P}_\theta = I_3 \dot{\theta} \]

\[ H = \frac{\text{P}_\phi^2}{2I_1} + \frac{\text{P}_\psi^2}{2I_3} + \frac{(\text{P}_\phi - \text{P}_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} + Mgh \cos \theta \]

\[ \equiv V(\theta) \]
1) notation

- $H$ does not depend on $\phi$ or $\psi$
- only dynamical variables in $H$: $p_\theta, \theta$

$\Rightarrow \theta$ coordinate performs one dimensional motion in effective potential $V(\theta)$

![Graph of $V(\theta)$]

2) possible motions

$E = E_0 = \min_{\theta} V(\theta) \rightarrow \theta$ constant $\rightarrow \phi$ constant $\rightarrow \psi$ constant

From maximum:

when steady: $\frac{d^2 \theta}{d\theta^2} = 0 \rightarrow$ quadratic equation for $p_\theta, P_y, c_\theta$

- only real solution if

$$\omega^2 \geq \frac{4M_\theta I_1 c_\theta}{I_3}$$

- need minimum angular velocity that depends on $\omega$

$\rightarrow$ MC #2