Two cases:

- Periodic motion: "closed orbit"
- Non-periodic motion: "open orbit"

How to decide?

\[ \Delta \Theta = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{2\mu}{\sqrt{2\mu(E-U(r)) - \frac{E^2}{2\mu^2}}} \, dr \]

\[ \Rightarrow \text{during one period of } x \text{-oscillations } \Theta \text{ change by } 2\Delta \Theta \]

\[ \Rightarrow \text{closed orbit } \Rightarrow \frac{\Delta \Theta}{\pi} < Q \]

\( E > 0 \): only one turning point, motion bounded from one side, "scattering"

### 3.1.3 Kepler's problem

Specifically \( U(r) = -\frac{k}{r} \)

\[ \Theta(r) = \int \frac{2\mu \, dr}{\sqrt{2\mu(E + \frac{k}{r} - \frac{E^2}{2\mu^2})}} \]

\[ = -\int \frac{du}{\sqrt{2\mu(E + \frac{k}{u} - \frac{E^2}{2\mu})}} \]

\[ = -\int \frac{du}{\sqrt{2\mu(E + \frac{k}{u^2} - \frac{(u - \frac{k}{\mu})^2}{2\mu})}} \]

\[ = -\frac{1}{\sqrt{2\mu(E + \frac{k}{u^2})}} \int \frac{du}{\sqrt{1 - \frac{(u - \frac{k}{\mu})^2}{2\mu(E + \frac{k}{u^2})}}} \]
\[ v = \frac{u - \frac{\mu k}{E}}{\sqrt{2\mu (E + \frac{\mu k^2}{2c^2})}} \Rightarrow \frac{dv}{u(1-u^2)} = \arccos(u) \]

\[ \cos \theta = v = \frac{\frac{v}{\sqrt{1 + \frac{2\mu k^2}{\mu k^2}}}}{\frac{v}{\sqrt{1 + \frac{2\mu k^2}{\mu k^2}}}} = \frac{\frac{v^2}{\mu k^2}}{\sqrt{1 + \frac{2\mu k^2}{\mu k^2}}} \]

Define \( e = \sqrt{1 + \frac{2\mu k^2}{\mu k^2}} \) "eccentricity"

\[ a = \frac{\alpha^2}{\mu k^2} \] 2a is "latus rectum"

\[ \frac{1}{e^2} = 1 + e \cos \theta \]

Diagram:
- \( E = 0 \): circle
- \( 0 < E < 1 \): ellipse
- \( E = 1 \): parabola
- \( E > 1 \): hyperbola

Kepler's First Law
Assume $e < 0 \implies$ elliptical orbits $\implies (E < 0)$

- What are the axes?

$$a = \frac{r_{\text{min}} + r_{\text{max}}}{2} = \frac{a}{1 + e} = \frac{\mu}{2E} = \frac{\mu}{2|E|}$$

$$a \sin \theta = \frac{a}{1 + e \cos \theta}$$

$$L = \sqrt{\frac{\mu}{a \sin \theta}}$$

$$g(\theta) = \frac{\sin \theta}{(1 + e \cos \theta)^2} \implies \cos \theta_0 = -e$$

$$L = \frac{\mu}{a \sin \theta} = \frac{a}{1 - e^2} = \frac{\mu}{(1 - e^2) \sqrt{2|E|}}$$

- What is the area?

$$A = \pi a L = \frac{\pi \mu L}{\sqrt{2|E|}}$$

- What is the period?

$$\frac{dA}{dt} = \frac{L}{2\mu} \implies T = \frac{2\mu}{L} A = \frac{\pi \mu \sqrt{\mu}}{L^{3/2}} |E|^{3/2}$$

$$T^2 = \pi^2 \frac{\mu^2 L^4}{|E|^3} = \frac{4\pi^2 \mu}{L^2} a^3$$

**Kepler's Third Law**

**Kepler's Laws:**

I. Planets move in elliptical orbits with the Sun at the focus.

II. The area per unit time swept out by the radius vector from the Sun to a planet is constant.

III. The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.