### Many body mechanics

#### Central force motion

#### Test integral of motion (fifth reading assignment)

1. Condition on center of mass
   - Two particles: $\mathbf{r}_1, \mathbf{r}_2$, \( U(r) \)
   - \( \mathbf{F} = \mathbf{F}_1 - \mathbf{F}_2 \)

2. Generalized coordinates
   - \( R = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 \)
   - \( L = \frac{1}{2} (m_1 + m_2) \dot{R}^2 + \frac{1}{2} \mu \dot{r}^2 - U(r) \)

- \( (\dot{R} \text{ cyclic} \Rightarrow \frac{\partial L}{\partial \dot{R}} \text{ constant } = (m_1 + m_2) \dot{R} \text{ constant} ) \)

- \( \frac{\partial L}{\partial r} = 0 \)

2) What is total velocity?

3) Why isn't conserved?

- Total angular symmetry \( \Rightarrow \) constant \( \Rightarrow \) motion plane \( \Rightarrow \) direction of \( \ddot{r} \)

\[ L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \]

\( \Rightarrow \) "cyclic" (i.e., \( L \) does not depend on \( \theta \)) \( \Rightarrow \frac{\partial L}{\partial \theta} = 0 \Rightarrow \ddot{\theta} = 0 \) constant
\( P_0 = \mu \gamma^2 \theta \quad (= L_2) \)

\[ \gamma = r \int \frac{dA}{d\theta} = \frac{1}{2} \gamma^2 \theta \]

\[ \frac{d\gamma}{dt} = \frac{1}{2} \gamma^2 2\dot{\theta} = \frac{3}{2} \gamma^2 \dot{\theta} = \frac{P_0}{2\mu} \]

4) integrals of motion

6 degree of freedom \rightarrow 6 coupled differential equation of second order \rightarrow 12 integration constants

Rate of mass least motion \rightarrow 3 coupled differential equation \rightarrow 6 integration constants

\((+R(0), \dot{R}(0))\)

relative motion plane \rightarrow 2 degree of freedom \( r, \theta \)

4 integration constants

2nd order ODE \rightarrow 1st order ODE \rightarrow 4th order ODE \rightarrow vehicles

no contact \rightarrow no integration constant, 1st order ODE for \( \theta \)

E = constant \rightarrow only integration constant, 1st order ODE for \( r \)

\[ H = \frac{1}{2} \mu \gamma^2 + \frac{1}{2} \frac{\dot{r}^2}{\mu \gamma^2} + U(r) = E \]

\[ \dot{r} = \pm \sqrt{ \frac{2}{\mu} (E - U(r)) - \frac{\dot{r}^2}{\mu \gamma^2} } \quad \text{first order ODE for } r \]

\[ \dot{\theta} = \frac{1}{\mu \gamma^2} \quad \text{(known) first order ODE for } \theta \quad \text{\( r(0) \) is known} \]
\[ \gamma = \pm \sqrt{\frac{2}{\mu} [E - U(r)] - \frac{\dot{r}^2}{r^2}} \]

\[ \Rightarrow \int dt = \pm \int \frac{dr}{\sqrt{\frac{2}{\mu} [E - U(r)] - \frac{\dot{r}^2}{r^2}}} \rightarrow t(r) - r(t) \]

\[ \dot{\theta}(t) = \frac{L}{\mu r^2(t)} \rightarrow \theta(t) \]

Really want to know \( \gamma(\theta) \) "orbit"

\[ \frac{dr}{d\theta} = \frac{dr}{dt} \frac{dt}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = \pm \frac{\dot{r}^2}{L} \frac{2}{\mu} [E - U(r)] - \frac{\dot{r}^2}{r^2} \]

\[ \Rightarrow \int d\theta = \int dr \frac{\dot{r}^2}{L \left( \frac{2}{\mu} [E - U(r)] - \frac{\dot{r}^2}{2r^2} \right)} \rightarrow \theta(r) \rightarrow \gamma(\theta) \]

Alternatively:

\[ L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \]

\[ \frac{\partial L}{\partial \dot{r}} = \mu \dot{r} \dot{\theta}^2 - \frac{\partial U}{\partial r} = \mu \dot{r} \dot{\theta}^2 + P(r) \]

\[ \dot{\theta} = \frac{L}{\mu r^2} \]

\[ \frac{\partial L}{\partial \dot{\theta}} = \mu \dot{r} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \mu \ddot{r} \]

\[ \Rightarrow \mu \ddot{r} = \frac{\dot{r}^2}{\mu r^3} + P(r) \]