Properties:
- Generalized coordinates do not have to have dimension length.
- For any system there are many ways to choose generalized coordinates, some lead to easier equations of motion than others.
- The kinetic energy in generalized coordinates is a function of the $q_i$ and the $\dot{q}_i$.

Equation of motion in generalized coordinates:
$$\frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial q_i} = 0 \quad \text{for } i = 1, \ldots, s$$

As long as:
1. Forces (the non-conservative) are derivable from a potential.
2. Constraints connect coordinates and possibly times, but not velocities by equalities ("holonomic constraints").

II. 2.5 Conservation laws

Symmetry of the problem:
\[ \rightarrow \text{Lagrangian has invariance (does not depend on certain variables)} \]
\[ \rightarrow \text{conservation laws} \]
Conservation of energy

Lagrangian time-independent \( \Rightarrow \frac{\partial L}{\partial t} = 0 \)

\[ \frac{dl}{dt} = \sum_{\alpha} \frac{\partial L}{\partial q_\alpha} \dot{q}_\alpha + \sum_{\alpha} \frac{\partial L}{\partial \dot{q}_\alpha} \ddot{q}_\alpha = \frac{dl}{dt} \sum_{\alpha} \frac{\partial L}{\partial q_\alpha} \dot{q}_\alpha \]

Lagrange's equations \( \frac{dl}{dt} \frac{\partial}{\partial q_\alpha} \)

Define \( H = \sum_{\alpha} \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L \) "Hamiltonian"

\[ \frac{dL}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt} H = 0 \quad \Rightarrow \quad H \text{ is a constant of the motion} \]

Assume:

1. The transformation \( x_{ai} (q_\alpha) \) do not depend on time
2. The potential \( U (q_\alpha, \dot{q}_\alpha, t) \) does not depend on \( \dot{q}_\alpha \)

\[ \frac{dl}{dt} = \frac{\partial (T-U) \sum \frac{\partial T}{\partial \dot{q}_\alpha}}{\partial \dot{q}_\alpha} \]

\[ T = \sum_{\alpha} \frac{1}{2} m_\alpha \dot{x}_{i,\alpha}^2 \]

\[ x_{i,\alpha} = \sum_{\alpha} \frac{\partial x_{ai}}{\partial q_\alpha} \dot{q}_\alpha \]

\[ T = \sum_{\alpha} \sum \frac{1}{2} m_\alpha \dot{x}_{i,\alpha}^2 \]

\[ \frac{dT}{dt} = \sum_{\alpha} \sum_{\alpha} m_\alpha \frac{\partial x_{ai}}{\partial q_\alpha} \frac{\partial x_{ai}}{\partial \dot{q}_\alpha} \dot{q}_\alpha \dot{q}_{i,\alpha} \]

\[ \Rightarrow \quad \sum_{\alpha} \dot{q}_\alpha \frac{dT}{dt} = \sum_{\alpha} \sum_{\alpha} m_\alpha \frac{\partial x_{ai}}{\partial q_\alpha} \frac{\partial x_{ai}}{\partial \dot{q}_\alpha} \dot{q}_\alpha \dot{q}_{i,\alpha} = 2T \]

\[ H = 2T - (T-U) = T + U \quad \text{energy} \]

(lbut only if 1. and 2. are true)
Conservation of (angular) momentum

In general

Physics is symmetric with respect to $q_i \rightarrow q_i + \epsilon$

$Lagrangian independent of $q_i$, "$q_i$ is cyclic"

$\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{dt}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$

$\frac{\partial L}{\partial \dot{q}_i}$ constant of the motion

Specifically cartesian coordinates

$L = T - U = \frac{1}{2} m \dot{x}_x^2 - U$

$\frac{\partial U}{\partial x_i} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}_i} = m \ddot{x}_i = \dot{p}_i$ constant

Many particles: $x_1, x_2, \ldots, x_n$

$\overline{y}_i^j = \overline{x}_i^j$, $\overline{y}_i^2 = \overline{x}_2 - \overline{x}_1$, \ldots, $\overline{y}_i^n = \overline{x}_n - \overline{x}_1$

$T = \sum_{a=1}^{n} \frac{1}{2} m_a \dot{x}_a^2 = \frac{1}{2} m_1 \dot{\overline{x}}_1^2 + \sum_{a=2}^{n} \frac{1}{2} m_a (\dot{\overline{y}}_a^1 + \dot{\overline{x}}_a^1)^2$

$\frac{\partial U}{\partial \dot{y}_i}$ (translational invariance)

$\Rightarrow \frac{\partial L}{\partial \dot{y}_i}$ constant

$\frac{\partial T}{\partial \dot{y}_i} = m_1 \ddot{x}_1 + \sum_{a=2}^{n} m_a (\dot{\overline{y}}_a^1 + \dot{\overline{x}}_a^1)$

$= m_1 \ddot{x}_1 + \sum_{a=2}^{n} m_a \ddot{\overline{x}}_a^1 = 2 \sum_{a=1}^{n} m_a \ddot{x}_a = p_i$
Specifically cylindrical coordinates

\[ x = r \cos \phi \]
\[ y = r \sin \phi \]
\[ z = z \]

\[ \dot{x} = \dot{r} \cos \phi - r \dot{\phi} \sin \phi \]
\[ \dot{y} = \dot{r} \sin \phi + r \dot{\phi} \cos \phi \]
\[ \dot{z} = \dot{z} \]

\[ T = \frac{1}{2} m \left( x^2 + y^2 + z^2 \right) = \frac{1}{2} m \left[ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right] \]

\[ = \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\phi}^2 + \dot{z}^2 \right) \]

\[ \frac{\partial U}{\partial \theta} = 0 \quad \text{(rotational invariance)} \]

\[ \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = \text{constant} \]

\[ m r^2 \dot{\theta} = m \left[ \left( \frac{\dot{r} \cos \phi - r \dot{\phi} \sin \phi}{z} \right) \times \left( \frac{\dot{r} \sin \phi + r \dot{\phi} \cos \phi}{z} \right) \right] = L \]

volatility of the choice of coordinate system:

U totally rotationally invariant \( \Rightarrow \) constant

II. 2.6 Canonical equations of motion

For a set of generalized coordinates, q_3 \( \text{def} \) generalized momenta as

\[ p_3 = \frac{\partial L}{\partial \dot{q}_3} \]

role for \( q_3 = q_3 \left( q_k, \pi_k, t \right) \)

\[ \Rightarrow \] can write Hamiltonian as function of \( q_k, \pi_k \) and t

\[ H = \sum_{\beta} p_\beta \dot{q}_\beta - L = H \left( q_k, \pi_k, t \right) \]