II.13 Euler's equation of the second kind

Euler's equation of the first kind:
\[
\frac{\partial f}{\partial y} - \frac{x}{\partial x} \frac{\partial f}{\partial y'} = 0
\]

Important special case: \( \frac{\partial f}{\partial x} = 0 \) (second order differential equation for \( y(x) \))

\[
\frac{d}{dx} f(y(x), y'(x), x) = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial x}
\]

\[
\frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right) = y'' \frac{\partial f}{\partial y''} + y' \frac{\partial f}{\partial y'}
\]

\[
= \frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial y} y' + y' \frac{\partial f}{\partial y'} \frac{\partial f}{\partial y'}
\]

\[
= \frac{d}{dx} \left[ f - y' \frac{\partial f}{\partial y'} \right] = 0
\]

\[
f - y' \frac{\partial f}{\partial y'} = \text{constant} \quad \text{for } \frac{\partial f}{\partial x} = 0
\]

(First order differential equation for \( y(x) \))
II. 1. 4 Function with several dependent variables

\[ y(x) \rightarrow \mathbf{y}(x) = (y_1(x), \ldots, y_n(x)) \]

\[ f(y, y', x) \rightarrow f(\mathbf{y}, \mathbf{y}', x) \]

\[ \mathcal{J}[\mathbf{y}(x)] = \int_{x_1}^{x_2} f(\mathbf{y}(x), \mathbf{y}'(x), x) \, dx \]

Maximum or minimum if for every \( \mathbf{\eta}(x) \) with \( \mathbf{\eta}(x) = 0 \)

\[ \mathbf{\eta}(x) \rightarrow \mathcal{J}[\mathbf{y}(x) + \epsilon \mathbf{\eta}(x)] \]

let, maximum / minimum if \( \epsilon = 0 \)

\[ \mathbf{\eta}(x) \text{ has } n \text{ independent direction} \rightarrow n \text{ independent equations} \]

\[ \frac{df}{dy_i} - \frac{d}{dx} \frac{df}{dy_i} = 0 \quad \text{for } i = 1, \ldots, n \]

system of ordinary second order differential equations