Variational problem:

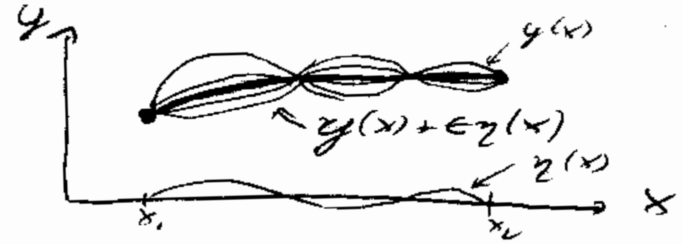
$$\boxed{\text{Find } y(x) \text{ that } \underset{(\text{or minimizes})}{\text{maximizes}}\ J[y(x)]\ !}$$

What does this mean?

For any (differentiable) $\eta(x)$ with $\eta(x_1) = \eta(x_2) = 0$

$$\epsilon \to J[y(x) + \epsilon\eta(x)] \qquad (\mathbb{R} \to \mathbb{R})$$

has its maximum at $\epsilon = 0$



"any deformation of $y(x)$ decreases J"

II.1.2 Euler's equation of the first kind

What is a necessary condition? "$\frac{dJ}{dy(x)} = 0$" ?

$\eta(x)$ arbitrary with $\eta(x_1) = \eta(x_2) = 0$

$$\Rightarrow \quad 0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} J[y(x) + \epsilon\eta(x)]$$

$$= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{x_1}^{x_2} f(y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x); x)\, dx$$

$$= \int_{x_1}^{x_2} \frac{d}{d\epsilon}\Big|_{\epsilon=0} f(y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x); x)\, dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}(y(x), y'(x), x)\,\eta(x) + \frac{\partial f}{\partial y'}(y(x), y'(x), x)\,\eta'(x)\right] dx$$

$$) \quad \underset{PI}{=} \int_{x_1}^{x_2} \frac{\partial f}{\partial y}(y(x), y'(x), x)\,\eta(x)\,dx + \underbrace{\frac{\partial f}{\partial y'}(y(x), y'(x), x)\,\eta(x)\Big|_{x_1}^{x_2}}_{=0 \text{ since } \eta(x_1)=\eta(x_2)=0} - \int_{x_1}^{x_2} \eta(x)\,\frac{d}{dx}\frac{\partial f}{\partial y'}(y(x), y'(x), x)\,dx$$

$$0 = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\eta(x)\,dx$$ for any $\eta(x)$ with $\eta(x_1)=\eta(x_2)=0$

$$\Rightarrow \quad \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$ Euler's equation of the first kind

$\downarrow$ 4/23