II. Lagrangian mechanics

- Mechanical systems often are subject to constraints that reduce number of degrees of freedom.

\[ 2 \text{ mass} \rightarrow 6 \text{ coordinates} \quad (n=30) \]
\[ 2 \text{ rods} \rightarrow 2 \text{ constraints} \rightarrow 4 + 6 - 2 \text{ degree of freedom} \]

- It is often cumbersome to calculate the resulting forces.

- Mechanical systems are often better described in other variables than \((x, y, z)\).

but: \[ \mathbf{F}=\mathbf{ma} \] requires forces and \((x, y, z)\)-description.

\[ \text{Wanted: Formalism that allows derivation of equations of motion using arbitrary variables and including constraints naturally.} \]

\[ \Rightarrow \text{Lagrangian formulation of mechanics} \]

II.1 Calculus of variation

II.1.1 The mathematical problem

Givin a function \( f(y, y', x) \) of three variables, study the functional

\[ J[y(x)] = \int_{x_1}^{x_2} f(y(x), y'(x); x) \, dx \]

maps \( y(x) \): function on interval \([x_1, x_2]\) into a real number.
Variational problem:

Find \( y(x) \) that maximizes \( J[y(x)] \).

What does this mean?

For any differentiable \( \eta(x) \) with \( \eta(x_i) = \eta(x_e) = 0 \),

\[
\eta(x) \to J[y(x) + \eta(x)] \quad (R \to R)
\]

has its maximum at \( \epsilon = 0 \).

"Any deformation of \( y(x) \) decreases \( J"."

Euler's equation of the first kind:

What is a necessary condition? \( \frac{\partial J}{\partial y(x)} = 0 \)?

\( \eta(x) \) arbitrary with \( \eta(x_i) = \eta(x_e) = 0 \)

\[
0 = \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} J[y(x) + \epsilon \eta(x)]
\]

\[
= \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} \int_{x_i}^{x_e} f(y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x); x) \, dx
\]

\[
= \int_{x_i}^{x_e} \frac{d}{d\epsilon} \bigg|_{\epsilon = 0} f(y(x) + \epsilon \eta(x), y'(x) + \epsilon \eta'(x); x) \, dx
\]

\[
= \int_{x_i}^{x_e} \left[ \frac{\partial f}{\partial y}(y(x), y'(x), x) \eta(x) + \frac{\partial f}{\partial y'}(y(x), y'(x), x) \eta'(x) \right] dx
\]
\[ \frac{d}{dx} \left( y(x), y'(x), x \right) \mathcal{L}(x) dx + \frac{df}{dy} \left( y(x), y'(x), x \right) \mathcal{L}(x) \bigg|_{x_1}^{x_2} = 0 \]

\[ \mathcal{L}(x) \frac{d}{dx} \frac{df}{dy} \left( y(x), y'(x), x \right) dx \]

\[ 0 = \int_{x_1}^{x_2} \left( \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy} \right) \mathcal{L}(x) dx \quad \text{for any } \mathcal{L}(x) \text{ with } \mathcal{L}(x) \mathcal{L}(x) = 0 \]

\[ \frac{df}{dy} - \frac{d}{dx} \frac{df}{dy} = 0 \]

Euler's equation of the first kind

Second order differential equation for \( \mathcal{L}(x) \).